Unit- III

2/6/2022
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Uniform Distribution:-

Let "x" is a continuous random variable & f(x) is probability density function of the distribution. then,

i) Mean of the distribution

$$\mu = E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

ii) The Median is M

- M divides the total area under the curve into equal parts.

$$\text{Median } M = \int_{-\infty}^{M} f(x)\, dx = \int_{M}^{\infty} f(x)\, dx = \frac{1}{2}$$

By solving the above equation we get 'M' which is the median of the distribution.

iii) The mode of the distribution is the value of x for which f(x) is maximum is $f'(x)=0$ and $f''(x)<0$ then value of x is mode of the distribution.

Uniform Distribution

A continuous random variable 'x' is said to follow the uniform distribution if its probability density function is given by

$$\boxed{\begin{aligned} f(x) &= \frac{1}{b-a} \quad ; \quad a<x<b \\ &= 0 \quad ; \quad \text{otherwise} \end{aligned}}$$

- In this distribution for different values of 'x' the probability is same.
- Uniform distribution is also called as <u>Rectangular distribution</u>.

Q. Find the moment generating function and cummulant generating function and also the characteristic function of uniform distribution.

Sol:-

Uniform distribution $f(x) = \frac{1}{b-a}$; $a<x<b$

$= 0$; otherwise

$$MGf = M_x(t) = E(e^{tx}) = \int_a^b e^{tx} f(x)\,dx$$

$$\Rightarrow \int_a^b e^{tx} \cdot \frac{1}{b-a}\,dx \Rightarrow \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$\Rightarrow \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b \Rightarrow \frac{1}{t(b-a)} \cdot e^{bt-at}$$

$$\boxed{MGf = M_x(t) = \frac{e^{t(b-a)}}{t(b-a)}}$$

$$CGf = K_x(t) = \log M_x(t) = \log \frac{e^{t(b-a)}}{t(b-a)}$$

$$\boxed{CGF = \log \frac{e^{bt} - e^{at}}{t(b-a)}}$$

$$Cf = \phi_x(t) = E(e^{itx}) = \int_a^b e^{itx} f(x)\,dx$$

$$\Rightarrow \int_a^b e^{itx} \cdot \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b e^{itx}\,dx$$

$$\Rightarrow \frac{1}{b-a}\left[\frac{e^{itx}}{it}\right]_a^b$$

$$\boxed{Cf = \phi_x(t) = \frac{e^{itb} - e^{ita}}{it(b-a)}}$$

Q) show that the rectangular distribution $f(x) = 1,\ 0 < x < 1$

mean $= \frac{1}{2}$, Variance $= \frac{1}{12}$.

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Sol:- $\text{Mean} = E(x) = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot 1 \cdot dx = \left[\frac{x^2}{2}\right]_0^1 \Rightarrow \frac{1}{2}$

$$\boxed{\text{Mean} = E(x) = \frac{1}{2}}$$

$$E(x^2) = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot 1 \cdot dx \Rightarrow \left[\frac{x^3}{3}\right]_0^1 \Rightarrow \frac{1}{3}$$

$$\text{Variance } V(x) = E(x^2) - [E(x)]^2$$

$$\Rightarrow \frac{1}{3} - \frac{1}{4} \Rightarrow \frac{4-3}{12} = \frac{1}{12}$$

$$\boxed{\text{Variance } V(x) = \frac{1}{12}}$$

Exponential Distribution:-

A continuous random variable is 'x' is said to follow exponential distribution if its probability density function is

$$f(x) = \lambda e^{-\lambda x}, \quad x>0, \ y>0$$
$$= 0, \quad \text{otherwise}$$

where λ is the parameter

Q. Find the moment generating function and cummulant generating function and also the characteristic function of exponential distribution.

Sol:- $MGF = M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} f(x)\,dx$

$$\Rightarrow \int_0^\infty e^{tx} \cdot \lambda e^{-\lambda x}\,dx = \lambda \int_0^\infty e^{tx} \cdot e^{-\lambda x}\,dx$$

$$\Rightarrow \lambda \int_0^\infty e^{-(\lambda - t)x}\,dx \Rightarrow \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^\infty$$

$$\Rightarrow \frac{\lambda}{-(\lambda-t)}\left[e^{-\infty} - e^{-0}\right] \Rightarrow \frac{\lambda}{-(\lambda-t)}[0-1] \Rightarrow \frac{\lambda}{\lambda-t}$$

$$\Rightarrow \frac{\cancel{\lambda}}{\cancel{\lambda}\left[1-\frac{t}{\lambda}\right]}$$

$$\boxed{MGF = \left[1-\frac{t}{\lambda}\right]^{-1}}$$

$$CGF = K_x(t) = \log M_x(t) = \log\left[1-\frac{t}{\lambda}\right]^{-1}$$

$$\boxed{CGF = -\log\left[1-\frac{t}{\lambda}\right]}$$

$$CF = \phi_x(t) = E(e^{itx}) \Rightarrow \int_0^\infty e^{itx}\cdot f(x)\,dx$$

$$CF = \int_0^\infty e^{itx}\cdot \lambda e^{-\lambda x}\,dx \Rightarrow \lambda\int_0^\infty e^{itx}\cdot e^{-\lambda x}\,dx$$

$$CF = \lambda\int_0^\infty e^{-(\lambda-it)x}\cdot dx$$

$$\Rightarrow \lambda\left[\frac{e^{-(\lambda-it)x}}{-(\lambda-it)}\right]_0^\infty \Rightarrow \frac{\lambda}{-(\lambda-it)}\left[e^{-\infty}-e^{-0}\right]$$

$$\Rightarrow \frac{\lambda}{-(\lambda-it)}[0-1] = \frac{\lambda}{(\lambda-it)} \Rightarrow \frac{\lambda}{\lambda\left[1-\frac{it}{\lambda}\right]}$$

$$\boxed{CF = \phi_x(t) = \left[1-\frac{it}{\lambda}\right]^{-1}}$$

Q. Find the mean & Variance of exponential distribution where $f(x) = \frac{1}{b}e^{-\frac{(x-a)}{b}}$, $x > a$

Sol :- Mean $= E(x) = \int_a^\infty x f(x)\,dx$

$$\Rightarrow \int_a^\infty x\cdot\frac{1}{b}e^{-\frac{(x-a)}{b}}\cdot dx \Rightarrow \frac{1}{b}\int_a^\infty x\cdot e^{-\frac{(x-a)}{b}}\cdot dx$$

$$\Rightarrow \frac{1}{b}\left[\frac{x\cdot e^{-\frac{(x-a)}{b}}}{-\frac{1}{b}} - (1)\frac{e^{-\frac{(x-a)}{b}}}{\left(-\frac{1}{b}\right)^2}\right]_a^\infty$$

$$\Rightarrow \frac{1}{b}\left\{0-0-\left[\frac{a}{\left(-\frac{1}{b}\right)}-\frac{1}{\left(\frac{1}{b}\right)^2}\right]\right\}$$

$$\Rightarrow \frac{1}{b}\left[ab+b^2\right]$$

$$\Rightarrow \boxed{\text{Mean}= E(x) = (a+b)}$$

$$E(x^2) = \int_a^\infty x^2 f(x)\,dx = \int_a^\infty x^2 \cdot \frac{1}{b} e^{\frac{-(x-a)}{b}} \cdot dx$$

$$\Rightarrow \frac{1}{b}\int_a^\infty x^2 \cdot e^{\frac{-(x-a)}{b}} \cdot dx$$

$$\Rightarrow \frac{1}{b}\left[\frac{x^2 \cdot e^{\frac{-(x-a)}{b}}}{-\frac{1}{b}} - \frac{2x\, e^{\frac{-(x-a)}{b}}}{\left(-\frac{1}{b}\right)^2} + \frac{2 \cdot e^{\frac{-(x-a)}{b}}}{\left(-\frac{1}{b}\right)^3}\right]_a^\infty$$

$$\Rightarrow \frac{1}{b}\left[-\left(\frac{a^2}{\left(-\frac{1}{b}\right)} - \frac{2a}{\left(\frac{1}{b^2}\right)} - \frac{2}{\left(\frac{1}{b^3}\right)}\right)\right]$$

$$\Rightarrow \frac{1}{b}\left[a^2 b + 2ab^2 + 2b^3\right]$$

$$E(x^2) = a^2 + 2ab + 2b^2$$

Variance $V(x) = E(x^2) - [E(x)]^2$

$$\Rightarrow a^2 + 2ab + 2b^2 - (a+b)^2$$

$$\Rightarrow a^2 + 2ab + 2b^2 - a^2 - 2ab - b^2$$

$$= b^2$$

$$\boxed{\text{Variance } V(x) = b^2}$$

Normal Distribution:-

If 'x' is a continuous random variable following normal probability distribution with mean 'μ' and standard deviation 'σ' then its probability distribution function is given by $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot dx$

where μ and σ are called parameters of normal distribution.

Put $z = \frac{x-\mu}{\sigma}$

$dz = \frac{dx}{\sigma}$

$\boxed{dx = \sigma \, dz}$

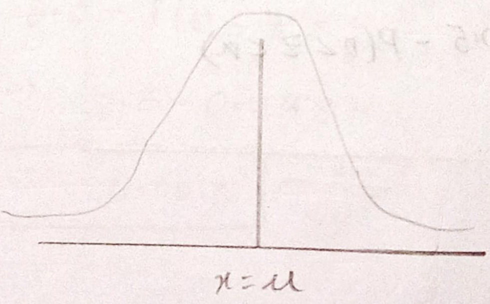
where 'z' is called the standard normal variable.

$\Rightarrow f(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \sigma \, dz$

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2} \cdot dz$.

Q. Write the basic properties of normal distribution.

Sol:-



→ The curve is symmetrical above the line $x = \mu$ $(z=0)$. It is ~~same~~ a bell shaped curve, it has the same shape on either side of the line $x = \mu$.

→ Mean, Median, and mode of the distribution coincides because the distribution is symmetrical and single peaked then

Mean = Median = Mode = μ.

→ No portion of the curve lies below the x-axis since $f(x)$ can't be negative.

→ Distribution is uni-model the only mode occuring at $x = \mu$.

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<u>Note:-</u>

1. $P(a<z<b) = P(0<z<b) - P(0<z<a)$
2. $P(-a<z<b) = P(0<z<b) + P(0<z<a)$
3. $P(z>a) = 0.5 - P(0<z<a)$
4. $P(z<a) = 0.5 + P(0<z<a)$
5. $P(z>-a) = 0.5 + P(0<z<a)$
6. $P(z<-a) = 0.5 - P(0<z<a)$

Q. If x is a normal variant, mean is 30 and S.D is 5. Find the probability that

i) $26 \le x \le 40$

ii) $x \ge 45$

Sol:- Mean $= \mu = 30$, S.D $= \sigma = 5$

$\boxed{z = \frac{x-\mu}{\sigma}} \Rightarrow \frac{x-30}{5}$

$\Rightarrow z_1 = \frac{26-30}{5} \Rightarrow -\frac{4}{5} = -0.8$

$z_2 = \frac{40-30}{5} = \frac{10}{5} = 2$

i) $P(26 \le x \le 40) = P(-0.8 \le z \le 2)$

$\Rightarrow P(0 \le z \le 0.8) + P(0 \le z \le 2)$

$\Rightarrow 0.2881 + 0.4772$

$\boxed{P(26 \le x \le 40) \Rightarrow 0.7653}$

ii) $P(x \ge 45)$

$z = \frac{45-30}{5} = \frac{15}{5} = 3$

$P(x \ge 45) = P(x \ge 3)$

$P(x \ge 45) \Rightarrow 0.5 - P(0 \le z \le 3)$

$P(x \ge 45) \Rightarrow 0.5 - 0.4987$

$\boxed{P(x \ge 45) \Rightarrow 0.0013}$

Q. The weekly wages of the workers in a factory are distributed normally with the mean ₹800 & S.D is 60. If there are 500 employees in the factory. How ~~much~~ many of them have wages ~~is Above ₹~~

i) Below ₹740

ii) Above ₹1000

iii) Between ₹700 and ₹900.

Sol:- Mean $= \mu = 800$, $S.D = \sigma = 60$

$$Z = \frac{x-\mu}{\sigma} \Rightarrow \frac{x-800}{60}$$

i) Below ₹740

$x = 740$

$$Z = \frac{740-800}{60} \Rightarrow Z = \frac{-60}{60} \Rightarrow \boxed{Z=-1}$$

$P(x<740) = P(Z<-1)$

$\Rightarrow 0.5 - P(0<Z<1)$

$\Rightarrow 0.5 - 0.3413$

$\Rightarrow 0.1587$

$\Rightarrow 0.1587 \times 500$

$\boxed{P(x<740) \Rightarrow 79.35}$

ii) Above ₹1000

$$Z = \frac{1000-800}{60} \Rightarrow Z = \frac{200}{60}$$

$\Rightarrow \boxed{Z = 3.33}$

$P(x>1000) = P(z>3.33)$

$\Rightarrow 0.5 - P(0<z<3.33)$

$\Rightarrow 0.5 - 0.4996$

$\Rightarrow 0.0004$

$\Rightarrow 0.0004 \times 500$

$\boxed{P(x>1000) = 0.2}$

iii) Between ₹700 and ₹900

$z_1 = \frac{700-800}{60} \Rightarrow \boxed{z_1 = -1.66}$

$z_2 = \frac{900-800}{60} \Rightarrow \boxed{z_2 = 1.66}$

$P(700 \le x \le 900)$

$\Rightarrow P(-1.66 \le z \le 1.66)$

$\Rightarrow P(0 \le z \le 1.66) + P(0 \le z \le 1.66)$

$\Rightarrow 2P(0 \le z \le 1.66)$

$\Rightarrow 2 \times 0.4525$

$\Rightarrow 0.905 \times 500$

$\boxed{P(700 \le x \le 900) = 452.5}$

Q. The height of 100 school children are normally distributed with mean 110cm & variance is 16cm. How many students would have heights

i) Above 95cm

ii) Below 90cm

iii) Between 100cm & 120cm

iv) Above 115cm.

Sol:- Mean $= \mu = 110cm$, Variance $= 16$, $S.D = \sqrt{16} = 4$

$$Z = \frac{x-\mu}{\sigma} = \frac{x-110}{4}$$

i) Above 95cm

$$Z = \frac{95-110}{4} \Rightarrow \frac{-15}{4}$$

$$\boxed{Z = -3.75}$$

$P(x > 95) = P(Z > -3.75)$

$\Rightarrow 0.5 + P(0 < Z < 3.75)$

$\Rightarrow 0.5 + 0.4999$

$\Rightarrow 0.9999$

$\Rightarrow 0.9999 \times 100$

$$\boxed{P(x > 95) = 99.99}$$

ii) Below 90cm

$z = \frac{90-110}{4} \Rightarrow \frac{-20}{4}$

$\boxed{z \Rightarrow -5}$

$P(x<90) = P(z<-5)$

$\Rightarrow 0.5 - P(0<z<5)$

$\Rightarrow 0.5 - 0.5 = 0$

$\boxed{P[x<90) = 0}$

iii) Between 100cm & 120cm

$z_1 = \frac{100-110}{4} = \frac{-10}{4} = -2.5$

$z_2 = \frac{120-110}{4} = \frac{10}{4} = 2.5$

$P(100 \leq z \leq 120)$

$\Rightarrow P(0 \leq z \leq 120) + P(0 \leq z \leq 100)$

$\Rightarrow P(0 \leq z \leq 2.5) + P(0 \leq z \leq 2.5)$

$\Rightarrow 2\,P(0 \leq z \leq 2.5)$

$\Rightarrow 2 \times 0.4938$

$\Rightarrow 0.9876$

$\Rightarrow 0.9876 \times 100 \rightarrow 98.76$

$\boxed{P(100 \leq z \leq 120) = 98.76}$

iv) Above 115cm

$$z = \frac{115-110}{4} = \frac{5}{4} = 1.25$$

$P(x > 115) = P(z > 1.25)$

$\Rightarrow 0.5 - P(0 < z < 1.25)$

$\Rightarrow 0.5 - 0.3944$

$\Rightarrow 0.1056$

$\Rightarrow 0.1056 \times 100 = 10.56$

$$\boxed{P(x > 115) = 10.56}$$