Unit-II

5/5/22
10:30AM

Binomial Distribution

Let there be 'n' independent trails in an experiment.

Let a random variable 'x' denotes the number of success in these 'n' trails.

Let 'p' be the probability of success and 'q' be the that of failure in a single trail so that $\boxed{p+q=1}$

Let the trails be independent & P be a constant for every trail.

The probability for 'r' success in 'n' trails

$P(x=r) = n_{c_r} p^r q^{n-r}$ —① where $p+q=1$ & $r=0,1,2 \dots n$

$(x+a)^n = n_{c_0} x^n + n_{c_1} x^{n-1} a + n_{c_2} x^{n-2} a^2 + \dots + n_{c_r} x^{n-r} a^r + \dots + n_{c_n} a^n$

$$\boxed{T_{r+1} = n_{c_r} x^{n-r} a^r}$$

The distribution eq① is called the Binomial probability distribution and 'x' is called the Binomial variable.

Note:-

1. $P(x=r)$ is usually written as $p(r)$
2. i) In a binomial distribution 'n' the no. of trails are finite.

2. ii) Each trail has only two possible outcomes usually called success & failure

2. iii) All the trails are independent.

2. iv) P is constant for all the trails

$$^nC_0 x^0 + {^nC_1} x^{n-1} a^1 + \cdots + {^nC_n} a^n = (x+a)^n$$

$$^nC_0 a^n + {^nC_1} a^{n-1} x^1 + \cdots + {^nC_n} x^n = (a+x)^n$$

Recurrence or Recurssion formula for the Binomial Distribution:-

In a binomial distribution

$$p(r) = {^nC_r}\, p^r q^{n-r} = \frac{n!}{(n-r)!\,r!} \cdot p^r q^{n-r}$$

$$p(r+1) = {^nC_{r+1}}\, p^{r+1} \cdot q^{n-r-1} = \frac{n!}{(n-r-1)!\,(r+1)!} \cdot p^{r+1} q^{n-r-1}$$

$$\frac{p(r+1)}{p(r)} = \frac{n!}{(n-r-1)!\,(r+1)!} \times p^{r+1} q^{n-r-1} \times \frac{(n-r)!\,r!}{n!\, p^r q^{n-r}}$$

$$\frac{p(r+1)}{p(r)} = \frac{(n-r)(n-r-1)!\,r!}{(n-r-1)!\,(r+1)\,r!} \cdot \frac{p}{q}$$

$$\frac{p(r+1)}{p(r)} = \frac{n-r}{(r+1)} \cdot \frac{p}{q}$$

$$\boxed{p(r+1) = \frac{n-r}{(r+1)} \cdot \frac{p}{q} p(r)}$$ which is requried recurrence formula. Applying this formula successively we can find $p(1), p(2)$ ------ etc if $p(0)$ is known.

Q) Find the mean and variance of the binomial distribution.

Sol:- The Binomial distribution

$$p(r) = n_{C_r} \cdot p^r \cdot q^{n-r}$$

$$\text{Mean} = \mu = \sum_{r=0}^{n} r \cdot p(r) = \sum_{r=0}^{n} r \cdot n_{C_r} p^r q^{n-r}$$

$$\Rightarrow \sum_{r=0}^{n} r \cdot n_{C_r} \cdot q^{n-r} \cdot p^r$$

$$\Rightarrow 0 + 1 \cdot n_{C_1} q^{n-1} p^1 + 2 \cdot n_{C_2} q^{n-2} p^2 + 3 \cdot n_{C_3} q^{n-3} p^3 + \cdots + n \cdot n_{C_n} q^{n-n} p^n$$

$$\Rightarrow n \cdot pq^{n-1} + 2 \cdot \frac{n(n-1)}{2!} q^{n-2} p^2 + \frac{3n(n-1)(n-2)}{3!} q^{n-3} p^3 + \cdots + n \cdot p^n$$

$$\Rightarrow npq^{n-1} + n(n-1) q^{n-2} p^2 + \frac{n(n-1)(n-2)}{2} q^{n-3} p^3 + \cdots + np^n.$$

$$\Rightarrow np[q^{n-1} + (n-1) q^{n-2} p + \frac{(n-1)(n-2)}{2} q^{n-3} p^2 + \cdots + p^{n-1}].$$

$$= np[(n-1)_{C_0} q^{n-1} + n-1_{C_1} q^{n-2} p^1 + n-1_{C_2} q^{n-3} p^2 + \cdots + n-1_{C_{n-1}} p^{n-1}]$$

$$\Rightarrow np [q+p]^{n-1} \quad \boxed{\text{where } p+q=1}$$

$$\Rightarrow np \cdot (1)^{n-1}$$

$$\boxed{\text{Mean} = \mu = np}$$

Hence the mean of the binomial distribution is $\boxed{mean = \mu = np}$

Variance:-

Variance $\sigma^2 = \sum_{r=0}^{n} r^2 p(r) - \mu^2$

$$\Rightarrow \sum_{r=0}^{n} [r + r(r-1)]\, p(r) - \mu^2$$

$$\Rightarrow \underline{\sum_{r=0}^{n} r p(r)} + \sum_{r=0}^{n} r(r-1) p(r) - \mu^2$$

$$\Rightarrow \mu + \sum_{r=0}^{n} r(r-1) p(r) - \mu^2$$

$$\Rightarrow \mu + \sum_{r=0}^{n} r(r-1)\, {}^{n}C_{r}\, q^{n-r} p^{r} - \mu^2$$

$$\Rightarrow \mu + 0 + 0 + 2\cdot 1\cdot {}^{n}C_{2}\, q^{n-2} p^{2} + 3\cdot 2\, {}^{n}C_{3}\, q^{n-3} p^{3} + \dots + n(n-1)\, {}^{n}C_{n}\, q^{n-n} p^{n} - \mu^2$$

$$\Rightarrow \mu + \left[2\cdot \frac{n(n-1)}{2}\cdot q^{n-2} p^{2} + 3\cdot 2\cdot \frac{n(n-1)(n-2)}{6} q^{n-3} p^{3} + \dots + n(n-1) p^{n}\right] - \mu^2$$

$$\Rightarrow \mu + n(n-1)p^2 \left[q^{n-2} + (n-2) q^{n-3} p^{1} + \dots + p^{n-2}\right] - \mu^2$$

$$\Rightarrow \mu + n(n-1)p^2 \left[{}^{n-2}C_{0}\, q^{n-2} + {}^{n-2}C_{1}\, q^{n-3} p^{1} + \dots + {}^{n-2}C_{n-2}\, p^{n-2}\right] - \mu^2$$

$$\Rightarrow \mu + n(n-1)p^2 [p+q]^{n-2} - \mu^2$$

where $p+q=1$
$\Rightarrow q = 1-p$

$$\Rightarrow \mu + n(n-1)p^2 (1)^{n-2} - \mu^2$$

$$\Rightarrow np + n(n-1)p^2 - n^2 p^2$$

$$\Rightarrow np\,[1 + (n-1)p - np]$$

$\Rightarrow np[1+\cancel{np}-p-\cancel{np}]$

$\Rightarrow np[1-p] = npq$

$\boxed{\therefore \text{ Variance } \sigma^2 = npq}$

standard deviation $= \sqrt{\text{variance}} = \sqrt{npq}$.

Similarly we can prove that

$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq}$ skewness & kurtosis

$\beta_2 = \frac{\mu_4}{\mu_2} = 3 - \frac{1 \cdot 1 - 6pq}{npq}$ sknewess & kurtosis.

Q) find the binomial distribution of the following data and find expected frequencies.

check back for similar problem.

x :	0	1	2	3	4	5	6	7
f :	7	6	19	35	30	23	7	1

Sol :-

x	f	fx
0	7	0
1	6	6
2	19	38
3	35	105
4	30	120
5	23	115
6	7	42
7	1	7

$n = 7$ $N = 128$ $\Sigma fx = 433$

Mean $= \frac{\Sigma fx}{N} = \frac{433}{128} = 3.38$

But

mean $= np = 3.38$

$p = \frac{3.38}{n} = \frac{3.38}{7} = 0.482$

$\boxed{p+q=1} \Rightarrow q = 0.52$

$\boxed{p = 0.482}$ $\boxed{q = 0.52}$ $\boxed{n=7}$

Binomial Distribution

$P(X=x) = n_{C_x}\, q^{n-x}\, p^{x}$

$P(x=0) = 7_{C_0}\,(0.52)^{7}\,(0.48)^{0} \Rightarrow 1\times 0.0102\times 1 \Rightarrow 0.013$

$\boxed{P(0) = 0.013}$

The Recurrence for binomial

$P(x+1) = \dfrac{n-x}{x+1}\, P(x) \cdot \dfrac{p}{q}$

x	$P(x+1) = \dfrac{n-x}{x+1} \cdot \dfrac{p}{q} \cdot P(x)$
0	$P(1) = \dfrac{7-0}{0+1} \cdot \dfrac{p}{q}\, P(0) = \dfrac{7}{1} \times \dfrac{0.48}{0.52} \times 0.013 = \boxed{0.0665}$
1	$P(2) = \dfrac{7-1}{1+1} \cdot \dfrac{0.48}{0.52} \times p(1) = \dfrac{6}{2} \cdot \dfrac{0.48}{0.52} \times 0.0665 = \boxed{0.1841}$
2	$P(3) = \dfrac{7-2}{2+1} \cdot \dfrac{0.48}{0.52} \times P(2) = \dfrac{5}{3} \cdot \dfrac{0.48}{0.52} \times 0.1841 = \boxed{0.2832}$
3	$P(4) = \dfrac{7-3}{3+1} \cdot \dfrac{0.48}{0.52} \cdot p(3) = \dfrac{4}{4} \cdot \dfrac{0.48}{0.52} \times 0.2832 = \boxed{0.2614}$
4	$P(5) = \dfrac{7-4}{4+1}\, \dfrac{0.48}{0.52} \cdot P(4) = \dfrac{3}{5}\, \dfrac{0.48}{0.52} \times 0.2614 = \boxed{0.1448}$
5	$P(6) = \dfrac{7-5}{5+1} \cdot \dfrac{0.48}{0.52} \cdot p(5) = \dfrac{2}{6} \times \dfrac{0.48}{0.52} \cdot 0.1448 = \boxed{0.0446}$
6	$P(7) = \dfrac{7-6}{6+1} \cdot \dfrac{0.48}{0.52}\, p(6) = \dfrac{1}{7} \times \dfrac{0.48}{0.52} \times 0.0446 = \boxed{0.0057}$

Theoritical frequencies $\{ f(x) = N \times p(x) \}$

x	$P(x)$ prob	Expected (Theortical) frequencies
0	0.0103	$f(0) = 128 \times 0.0103 = 1$
1	0.0665	$f(1) = 128 \times 0.0665 = 9$
2	0.1841	$f(2) = 128 \times 0.1841 = 24$
3	0.2832	$f(3) = 128 \times 0.2832 = 36$
4	0.2614	$f(4) = 128 \times 0.2614 = 33$
5	0.1448	$f(5) = 128 \times 0.1448 = 18$
6	0.0446	$f(6) = 128 \times 0.0446 = 06$
7	0.0057	$f(7) = 128 \times 0.0057 = 01$

$\boxed{N = 128}$

Poission Distribution:-

A random variable 'x' is said to follow poission distribution if its probability density function is given by

$$P(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^{x}}{x!}, \quad \lambda = 0$$

$$= 0, \quad \text{otherwise}$$

where 'λ' is known as the parameter of distribution.

Q) show that mean and Variance of poisson distribution are equal.

Sol:- Poisson Distribution

$$P(x) = \sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\text{Mean} = \mu = \sum_{x=0}^{\infty} x \cdot p(x) \Rightarrow \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow e^{-\lambda} \sum_{x=1}^{\infty} \frac{x \cdot \lambda^x}{x(x-1)!} \Rightarrow e^{-\lambda} \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$\Rightarrow e^{-\lambda} \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots + \infty\right]$$

$$\Rightarrow e^{-\lambda} \lambda e^{\lambda}$$

$$\boxed{\text{Mean} = \mu = \lambda}$$

$$\boxed{\text{Variance} \quad V(x) = E(x^2) - [E(x)]^2}$$

$$E(x^2) = \sum_{x=0}^{\infty} x^2 p(x) = \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow \sum_{x=0}^{\infty} [x(x-1) + x] \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow \sum_{x=0}^{\infty} \frac{x(x-1)e^{-\lambda}\lambda^x}{x(x-1)(x-2)!} + \lambda$$

$$\Rightarrow e^{-\lambda}\lambda^2 \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda$$

$$\Rightarrow e^{-\lambda}\lambda^2\left[1+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots+\infty\right]+\lambda$$

$$\Rightarrow e^{-\lambda}\lambda^2 e^{\lambda}+\lambda$$

$$\boxed{E(x^2)=\lambda^2+\lambda}$$

Variance $V(x) = E(x^2) - [E(x)]^2$

$$\Rightarrow \lambda^2+\lambda-\lambda^2 = \boxed{\lambda}$$

Therefore,

$$\boxed{\text{Mean} = \text{Variance} = \lambda}$$

Hence, in poisson distribution mean & variance are equal.

Note:-

- Moment generating function

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx}\cdot p(x)$$

- Cummulant generating function

$$K_x(t) = \log M_x(t)$$

- characteristic function

$$\phi_x(t) = E(e^{+itx}) = \sum_{x=0}^{\infty} e^{+itx}\cdot p(x)$$

Q] Derive moment generating function & cummulant generating function of poisson distribution.

Sol:- Moment generating function

$$M_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} \cdot p(x)$$

$$\Rightarrow \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \lambda^x}{x!} \Rightarrow e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t \lambda)^x}{x!}$$

$$\Rightarrow e^{-\lambda} \left[1 + \frac{(e^t\lambda)^1}{1!} + \frac{(e^t\lambda)^2}{2!} + \frac{(e^t\lambda)^3}{3!} + \cdots + \infty\right]$$

$$\Rightarrow e^{-\lambda} e^{e^t \lambda} \Rightarrow e^{-\lambda + e^t \lambda}$$

$$\boxed{M_x(t) = e^{\lambda[e^t - 1]}}$$

Cummulant generating function

$$K_x(t) = \log M_x(t)$$

$$K_x(t) = \log e^{\lambda[e^t - 1]}$$

$$K_x(t) = \lambda(e^t - 1) \log_e e$$

$$\boxed{K_x(t) = \lambda[e^t - 1]}$$

Q] Derive characteristic function of poisson distribution.

Sol:- characteristic function

$$\phi_x(t) = E(e^{itx}) = \sum_{x=0}^{\infty} e^{itx} p(x)$$

$$\Rightarrow \sum_{x=0}^{\infty} e^{itx} \cdot \frac{e^{-\lambda} \lambda^x}{x!} \rightarrow e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{it}\lambda)^x}{x!}$$

$$\Rightarrow e^{-\lambda} \left[1 + \frac{(e^{it}\lambda)^1}{1!} + \frac{(e^{it}\lambda)^2}{2!} + \text{-----} \infty\right]$$

$$\Rightarrow e^{-\lambda} e^{e^{it} \cdot \lambda} \Rightarrow e^{-\lambda + e^{it} \cdot \lambda}$$

$$\boxed{\phi_x(t) = e^{\lambda[e^{it} - 1]}}$$

Q] If X is a poisson variable with P(X = 1) = 4P(X=2). find its mean

Sol:- Poisson distribution $P(X) = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!}$

$$P(x=1) = 4P(x=2)$$

$$\frac{e^{-\lambda} \lambda^1}{1!} = 4 \cdot \frac{e^{-\lambda} \lambda^2}{2!}$$

$$\lambda = 2\lambda^2$$

$$1 = 2\lambda$$

$$\lambda = \frac{1}{2}$$

$$\boxed{\text{Mean} = \frac{1}{2}}$$

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Q] Fit the poisson distribution to the following data and obtain the theoretical frequencies.

x :	0	1	2	3	4	5
f :	142	156	69	27	5	1

Sol:-

x	f	fx
0	142	0
1	156	156
2	69	138
3	27	81
4	5	20
5	1	5

$N = \Sigma f = 400$ $\quad$ $\Sigma fx = 400$

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{400}{400} = 1$$

$$\boxed{\lambda = 1}$$

$$\boxed{P(x) = \frac{e^{-\lambda}\lambda^x}{x!}} \Rightarrow \frac{e^{-1}1^x}{x!}$$

where $x = 0, 1, 2, 3, 4, 5 \ldots$

$$P(0) = \frac{e^{-1}1^0}{0!} = e^{-1} = \boxed{0.3678}$$

$$P(1) = \frac{e^{-1}1^1}{1!} = e^{-1} = \boxed{0.3678}$$

$$P(2) = \frac{e^{-1}1^2}{2!} = \frac{e^{-1}}{2} = \boxed{0.1839}$$

$$P(3) = \frac{e^{-1}1^3}{3!} = \frac{e^{-1}}{6} = \boxed{0.0613}$$

$$P(4) = \frac{e^{-1}1^4}{4!} = \frac{e^{-1}}{24} = \boxed{0.0153}$$

$$P(5) = \frac{e^{-1}1^5}{5!} = \frac{e^{-1}}{120} = \boxed{0.0030}$$

Theoretical frequencies $\{f(x) = N \cdot P(x)\}$

x	$f(x)$	Expected (Theoretical) frequency
0	0.3678	$f(0) = 400 \times 0.3678 = 147$
1	0.3678	$f(1) = 400 \times 0.3678 = 147$
2	0.1839	$f(2) = 400 \times 0.1839 = 74$
3	0.0613	$f(3) = 400 \times 0.0613 = 25$
4	0.0153	$f(4) = 400 \times 0.0153 = 6$
5	0.0030	$f(5) = 400 \times 0.0030 = 1$

$\boxed{N = 400}$

Q) fit the poisson distribution to the following data & obtain the theoretical frequencies.

x:	0	1	2	3	4	5
f:	122	60	69	27	5	1

Sol:-

x	f	fx
0	122	0
1	60	60
2	69	138
3	27	81
4	5	20
5	1	5

$N = \Sigma f = 284$ $\quad \Sigma fx = 304$

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{304}{284}$$

$\lambda = 1.07$

$$\boxed{P(x) = \frac{e^{-\lambda} \lambda^x}{x!}} = \frac{e^{-1.07} \cdot 1^x}{x!}$$

where

$x = 0, 1, 2, 3, 4, 5 \ldots$

$$P(0) = \frac{e^{-1.07}(1.07)^0}{0!} = \boxed{0.3430}$$

$$P(1) = \frac{e^{-1.07}(1.07)^1}{1!} = \boxed{0.3670}$$

$$P(2) = \frac{e^{-1.07}(1.07)^2}{2!} \Rightarrow \frac{0.3927}{2} = \boxed{0.1963}$$

$$P(3) = \frac{e^{-1.07}(1.07)^3}{3!} = \boxed{0.0700}$$

$$P(4) = \frac{e^{-1.07}(1.07)^4}{4!} = \boxed{0.01873}$$

$$P(5) = \frac{e^{-1.07}(1.07)^5}{5!} = \boxed{0.0040}$$

Theoretical frequencies $\{f(x) = N \cdot P(x)\}$

x	$f(x)$	Expected (Theoretical) frequency
0	0.3430	$f(0) = 284 \times 0.3430 = 98$
1	0.3670	$f(1) = 284 \times 0.3670 = 104$
2	0.1963	$f(2) = 284 \times 0.1963 = 56$
3	0.0700	$f(3) = 284 \times 0.0700 = 20$
4	0.0187	$f(4) = 284 \times 0.0187 = 5$
5	0.0040	$f(5) = 284 \times 0.0040 = 1$

$\boxed{N = 284}$

Q) Fit a poisson distribution to the following data and obtain the theoretical frequencies.

x:	0	1	2	3	4	5	6
f:	275	72	30	7	5	2	1

Sol:

x	f	fx
0	275	0
1	72	72
2	30	60
3	7	21
4	5	20
5	2	10
6	1	6

$N = \Sigma f = 392$ $\quad \Sigma fx = 189$

$$\text{Mean} = \frac{\Sigma fx}{\Sigma f} = \frac{189}{392}$$

$$\boxed{\lambda = 0.4821}$$

$$\boxed{P(x) = \frac{e^{-\lambda} \lambda^x}{x!}}$$

$$\Rightarrow \frac{e^{-0.48} (0.48)^x}{x!}$$

where $x = 0, 1, 2, 3, 4, 5, 6 \ldots$

$$P(0) = \frac{e^{-0.48} (0.48)^0}{0!} = \boxed{0.6187}$$

$$P(1) = \frac{e^{-0.48} (0.48)^1}{1!} = \boxed{0.2970}$$

$$P(2) = \frac{e^{-0.48} (0.48)^2}{2!} = \boxed{0.0712}$$

$$P(3) = \frac{e^{-0.48} (0.48)^3}{3!} = \boxed{0.0114}$$

$$P(4) = \frac{e^{-0.48} (0.48)^4}{4!} = \boxed{0.0013}$$

$P(5) = \dfrac{e^{-0.48}(0.48)^5}{5!} = \boxed{0.00013}$

$P(6) = \dfrac{e^{-0.48}(0.48)^6}{6!} = \boxed{0.00001}$

Theoretical frequencies $\{f(x) = N \cdot P(x)\}$

x	$f(x)$	Expected (Theoretical) frequency
0	0.6187	$f(0) = 242.5$
1	0.2970	$f(1) = 116.4$
2	0.0712	$f(2) = 27.9$
3	0.0114	$f(3) = 4.46$
4	0.0013	$f(4) = 0.1$
5	0.0013	$f(5) = 0.05$
6	0.0001	$f(6) = 0.00392$

$\boxed{N = 392}$