

# Laplace Transform :-

Definition : If  $f(t)$  is a function of 't' then its Laplace is given as

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = f(s)$$

Variable

Properties of L.T :-

1,  $\mathcal{L}\{c f(t)\} = c \mathcal{L}\{f(t)\} = c f(s)$

2,  $\mathcal{L}\{f(t) \pm g(t)\} = \mathcal{L}\{f(t)\} \pm \mathcal{L}\{g(t)\} = f(s) \pm g(s)$

Formulae :-

1,  $\mathcal{L}\{1\} = \frac{1}{s}$

2,  $\mathcal{L}\{t\} = \frac{1}{s^2}$

3,  $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$

4,  $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

5,  $\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}$

6,  $\mathcal{L}\{\cos at\} = \frac{s}{s^2 + a^2}$

7,  $\mathcal{L}\{\sinh at\} = \frac{a}{s^2 - a^2}$

8,  $\mathcal{L}\{\cosh at\} = \frac{s}{s^2 - a^2}$



19 Find the Laplace Transform of the following

$$e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$$

Soln Let

$$L\{f(t)\} = e^{2t} + 4t^3 - 2\sin 3t + 3\cos 3t$$

Take L.T on b.s

$$L\{f(t)\} = L\{e^{2t}\} + 4L\{t^3\} - 2L\{\sin 3t\} + 3L\{\cos 3t\}$$

$$f(s) = \frac{1}{s-2} + 4 \frac{3!}{s^3+1} - 2 \frac{3}{s^2+3^2} + 3 \frac{s}{s^2+3^2}$$

$$f(s) = \frac{1}{s-2} + \frac{24}{s^4} + \frac{3s-6}{s^2+9}$$

20

Find the L.T of  $\sin 2t \cdot \cos 3t$

Soln

$$f(t) = \sin 2t \cdot \cos 3t$$

$$f(t) = \frac{1}{2} 2 \cos 3t \sin 2t$$

$$f(t) = \frac{1}{2} [\sin 5t - \sin t]$$

Take L.T (O.B.S)

$$L\{f(t)\} = \frac{1}{2} L\{\sin 5t - \sin t\}$$

$$f(s) = \frac{1}{2} \left[ \frac{5}{s^2+25} - \frac{1}{s^2+1} \right]$$

$$= \frac{1}{2} \left[ \frac{5s^2+5-s^2-25}{(s^2+25)(s^2+1)} \right]$$

$$= \frac{1}{2} \left[ \frac{4s^2-20}{(s^2+25)(s^2+1)} \right]$$

$$= \frac{2s^2-10}{(s^2+25)(s^2+1)}$$

$$2 \cos A \sin B = \sin(A+B) - \sin(A-B)$$



3Q Find the Laplace Transform of  $\cos^3(2t)$

sol

$$f(t) = \cos^3(2t)$$
$$= \frac{\cos 3(2t) + 3 \cos(2t)}{4}$$

$$= \frac{1}{4} [\cos 6t + 3 \cos 2t]$$

Take L.T. O.B.S

$$2\{f(t)\} = \frac{1}{4} L\{\cos 6t + 3 \cos 2t\}$$

$$f(s) = \frac{1}{4} \left[ \frac{s}{s^2+6^2} + 3 \cdot \frac{s}{s^2+2^2} \right]$$
$$= \frac{1}{4} \left[ \frac{4s^3+112s}{(s^2+36)(s^2+4)} \right]$$

$$= \frac{s^3+28s}{(s^2+36)(s^2+4)}$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$
$$\frac{\cos 3\theta + 3\cos\theta}{4} = \cos^3\theta$$

3Q Find the LT of  $\sin^3 2t$

sol

$$\text{let } f(t) = \sin^3 2t$$
$$= \frac{1}{4} [3 \sin 2t - \sin 3(2t)]$$

Take L.T. O.B.S

$$2\{f(t)\} = \frac{1}{4} L\{3 \sin 2t - \sin 6t\}$$

$$f(s) = \frac{1}{4} \left[ 3 \frac{2}{s^2+2^2} - \frac{6}{s^2+6^2} \right]$$

$$= \frac{6}{4} \left[ \frac{1}{s^2+2^2} - \frac{1}{s^2+6^2} \right]$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$
$$4\sin^3\theta = 3\sin\theta - \sin 3\theta$$
$$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$



$$= \frac{b}{4} \left[ \frac{s^2 + 3b - s^2 - 4}{(s^2 + 4)(s^2 + 3b)} \right]$$

$$= \frac{b}{4}$$

I<sup>st</sup> Shifting Theorem.

Statement: If  $\mathcal{L}\{f(t)\} = f(s)$  then  $\mathcal{L}\{e^{at} f(t)\} = f(s-a)$

Proof: H.K.T

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt = f(s)$$

$$\mathcal{L}\{e^{at} f(t)\} = \int_0^{\infty} e^{-st} e^{at} f(t) dt$$

$$= \int_0^{\infty} e^{-st+at} f(t) dt$$

$$= \int_0^{\infty} e^{-(s-a)t} f(t) dt = f(s-a)$$

$$\therefore \mathcal{L}\{e^{at} f(t)\} = f(s-a)$$

Ex  
Exn

Find Laplace Transform of  $\mathcal{L}\{e^{-4t} t^2\}$

$$\text{Let } f(t) = t^2$$

Take L.T.O.B.s

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} = \frac{2!}{s^2+1} = \frac{2}{s^3} = f(s)$$

By I-shifting Theorem

$$\mathcal{L}\{e^{at} f(t)\} = f(s-a)$$

$$\mathcal{L}\{e^{-4t} t^2\} = \frac{2}{s^3 - 4}$$

Leave  $e^{at}$   
Take  $f(t)$

Take  $t^2$   
And L.T.

$s^3$

$s-a$

$s \rightarrow s-a$



20

Find the L.T of  $2\{(t+1)^2 e^t\}$

soln

$$\text{Let } f(t) = (t+1)^2 = t^2 + 2t + 1.$$

Take L.T on b.s

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2 + 2t + 1\}$$

$$= \frac{2!}{s^{2+1}} + 2 \frac{1}{s^2} + \frac{1}{s}$$

$$= \frac{2 + 2s + s^2}{s^3} = f(s)$$

By I shifting theorem

$$\mathcal{L}\{e^{at} f(t)\} = f(s-a)$$

$$\mathcal{L}\{e^t (t+1)^2\} = \frac{2 + 2(s-1) + (s-1)^2}{(s^2-1)^3}$$

20

Find  $\mathcal{L}\{e^t \sin t \cos t\}$

soln

$$\text{Let } f(t) = \sin t \cos t$$

$$= \frac{1}{2} \sin 2t$$

Take L.T on b.s

$$\mathcal{L}\{f(t)\} = \frac{1}{2} \mathcal{L}\{\sin 2t\} = \frac{1}{2} \frac{2}{s^2 + 2^2} = \frac{1}{s^2 + 4} = f(s)$$

By I-shifting theorem

$$\mathcal{L}\{e^{at} f(t)\} = f(s-a)$$

$$\mathcal{L}\{e^t \sin t \cos t\} = \frac{1}{(s-1)^2 + 4}$$



49

Find the Laplace Transform of  $e^{-3t} [2 \cos 5t - 3 \sin 5t]$

Soln

$$\text{Let } f(t) = 2 \cos 5t - 3 \sin 5t$$

Take L.T. on b.s

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{2 \cos 5t - 3 \sin 5t\} \\ &= 2 \mathcal{L}\{\cos 5t\} - 3 \mathcal{L}\{\sin 5t\} \\ &= 2 \cdot \frac{s}{s^2 + 25} - 3 \cdot \frac{5}{s^2 + 25} \end{aligned}$$

$$\mathcal{L}\{f(t)\} = \frac{2s - 15}{s^2 + 25} = f(s)$$

By 1<sup>st</sup> shifting theorem

$$\mathcal{L}\{e^{at} f(t)\} = f(s-a)$$

$$\begin{aligned} \mathcal{L}\{e^{-3t} 2 \cos 5t - 3 \sin 5t\} &= \frac{2(s+3) - 15}{(s+3)^2 + 25} \\ &= \frac{2s - 9}{s^2 + 6s + 34} \end{aligned}$$

50

Find the Laplace Transform of  $\mathcal{L}\{e^{-2t} \sin 3t \cdot \sin 2t\}$

Sol

$$\text{Let } f(t) = \sin 3t \cdot \sin 2t$$

$$\therefore f(t) = \frac{1}{2} 2 \sin 3t \sin 2t$$

$$f(t) = \frac{1}{2} [\cos t - \cos 5t]$$

Take L.T. on b.s

$$\mathcal{L}\{f(t)\} = \frac{1}{2} \mathcal{L}\{\cos t - \cos 5t\}$$

$$= \frac{1}{2} \left[ \frac{s}{s^2 + 1} - \frac{s}{s^2 + 25} \right]$$

$$= \frac{s}{2} \left[ \frac{s^2 + 25 - s^2 - 1}{(s^2 + 1)(s^2 + 25)} \right]$$



$$= \frac{s}{2} \left[ \frac{24}{(s^2+1)(s^2+25)} \right]$$

$$= \frac{12s}{(s^2+1)(s^2+25)}$$

By 1st shifting theorem

$$\mathcal{L}\{e^{at}f(t)\} = f(s-a)$$

$$\mathcal{L}\{e^{-2t} \sin t \cdot \sin 5t\} = \frac{12(s+2)}{[(s+2)^2+1][(s+2)^2+25]}$$

$$= \frac{12(s+2)}{(s^2+4s+4+1)(s^2+4s+29)}$$

69 Find Laplace Transform of  $\mathcal{L}\{\sinh at \cdot \cosh bt\}$

sol<sup>n</sup> Let  $f(t) = \sinh at \cdot \cosh bt$

$$= \left[ \frac{e^{-at}}{2} - \frac{e^{at}}{2} \right] \cosh bt$$

$$= \frac{1}{2} [e^{-at} \cosh bt - e^{at} \cosh bt]$$

By 1st shifting theorem

$$\mathcal{L}\{e^{at}f(t)\} = f(s-a)$$

$$f(s) = \frac{1}{2} \left[ \frac{s+a}{(s+a)^2+b^2} - \frac{s-a}{(s-a)^2+b^2} \right] \sinh^2$$

$$\sinh^2 x = \frac{e^x - e^{-x}}{2}$$

$$\cosh^2 x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$



$$\textcircled{1} \quad \mathcal{L}\{e^{at} \sin bt\} = \frac{b}{(s-a)^2 + b^2}$$

$$\textcircled{2} \quad \mathcal{L}\{e^{at} \cos bt\} = \frac{(s-a)}{(s-a)^2 + b^2}$$

$$\textcircled{3} \quad \mathcal{L}\{e^{at} \sinh bt\} = \frac{b}{(s-a)^2 - b^2}$$

$$\textcircled{4} \quad \mathcal{L}\{e^{at} \cosh bt\} = \frac{(s-a)}{(s-a)^2 - b^2}$$

4th mat

## II Shifting Theorem

Statement: If  $\mathcal{L}\{f(t)\} = f(s)$  &  $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then  $\mathcal{L}\{g(t)\} = e^{-as} f(s)$

~~$\mathcal{L}\{g(t)\}$  where  $g$~~

Proof

Left hand problem

Ex 1 Find the L.T of  $3t^2 - \cos 2t$

Sol

let  $f(t) = 3t^2 - \cos 2t$

Take L.T. on both sides

$$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{t^2\} - \mathcal{L}\{\cos 2t\}$$

$$f(s) = 3 \frac{2!}{s^2+1} - \frac{s}{s^2+2^2}$$

$$= \frac{6}{s^3} - \frac{s}{s^2+4}$$

Q Find L.T of  $\cosh at - \cos at$

Sol

Let  $f(t) = \cosh at - \cos at$

Take L.T. on both side

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cosh at\} - \mathcal{L}\{\cos at\}$$

$$= \frac{s}{s^2-a^2} - \frac{s}{s^2+a^2}$$

$$= \frac{s(s^2+a^2) - s(s^2-a^2)}{(s^2-a^2)(s^2+a^2)}$$

$$= \frac{s^3 + sa^2 - s^3 + sa^2}{s^4 + s^2a^2 - a^2s^2 - a^4}$$

$$= \frac{2sa^2}{s^4 - a^4}$$



10 Find LT of  $g(t)$  where  $g(t) = \begin{cases} \sin(t - 2\pi/3) & \forall t > 2\pi/3 \\ 0 & \forall t < 2\pi/3 \end{cases}$

58 let  $f(t) = \sin t$  &  $a = 2\pi/3$

Take LT on b.s

$$L\{f(t)\} = L\{\sin t\} = \frac{1}{s^2 + 1} = f(s)$$

$$\text{Given } g(t) = \begin{cases} \sin t - 2\pi/3 & \forall t > 2\pi/3 \\ 0 & \forall t < 2\pi/3 \end{cases}$$

By 2<sup>nd</sup> shifting theorem

$$L\{g(t)\} = e^{-as} f(s)$$

$$L\{g(t)\} = e^{-2\pi/3 s} \frac{1}{s^2 + 1} = \frac{e^{-2\pi/3 s}}{s^2 + 1}$$

20 Find LT  $g(t) = \begin{cases} \cos(t - 2\pi/3) & \forall t \geq 2\pi/3 \\ 0 & \forall t < 2\pi/3 \end{cases}$

58 let  $f(t) = \cos t$   $a = 2\pi/3$

Take LT on b.s

$$L\{f(t)\} = L\{\cos t\} = \frac{s}{s^2 + 1} = f(s) \checkmark$$

$$\text{Given } g(t) = \begin{cases} \cos(t - 2\pi/3) & \forall t > 2\pi/3 \\ 0 & \forall t < 2\pi/3 \end{cases}$$

By using 2<sup>nd</sup> shifting theorem

$$L\{g(t)\} = e^{-as} f(s)$$

$$L\{g(t)\} = e^{-2\pi/3 s} \frac{s}{s^2 + 1}$$

$$= \frac{e^{-2\pi/3 s} s}{s^2 + 1}$$



3Q Find L.T  $g(t) = \begin{cases} e^{t-a} & ; \forall t > a \\ 0 & ; \forall t < a \end{cases}$

soln Let  $f(t) = e^t$   
T.L.T on b.s

$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^t\} = \frac{1}{s-1} = f(s)$

Given  $g(t) = \begin{cases} e^{t-a} & ; \forall t > a \\ 0 & ; \forall t < a \end{cases}$

By 2<sup>nd</sup> shifting theorem

$\mathcal{L}\{g(t)\} = e^{-as} f(s)$

$\mathcal{L}\{g(t)\} = e^{-as} \frac{1}{s-1}$   
 $= \frac{e^{-as}}{s-1}$

5<sup>th</sup> April

Change of scale of property:

Note: Leave 'a' take remaining  $f(t)$

If  $\mathcal{L}\{f(t)\} = f(s)$  then L.T of  $\mathcal{L}\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

1Q Find L.T of  $\frac{1}{4} \cos 4t$  by using change of scale of property:

soln  $f(t) = \cos t$   
T.L.T on b.s

$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos t\} = \frac{s}{s^2+1} = f(s)$

By using C.S.P

$\mathcal{L}\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

$\mathcal{L}\{\cos 4t\} = \frac{1}{4} \cdot \frac{s/4}{\left(\frac{s}{4}\right)^2 + 1} = \frac{1}{4} \cdot \frac{s}{\frac{s^2+16}{16}} = \frac{s}{s^2+16}$



2a Find  $\mathcal{L}\{\sinh 2t\}$  by using change of scale of property.

let  $f(t) = \sinh t$

T.L.T on b.s

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sinh t\} = \frac{1}{s^2 - 1^2}$$

By C.S.P

$$\mathcal{L}\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$$

$$\mathcal{L}\{\sinh 2t\} = \frac{1}{2} \frac{1}{\left(\frac{s}{2}\right)^2 - 1} = \frac{1}{2} \frac{1}{\frac{s^2 - 4}{4}} = \frac{4}{2(s^2 - 4)} = \frac{2}{s^2 - 4}$$

$$\therefore \mathcal{L}\{\sinh 2t\} = \frac{2}{s^2 - 4}$$

$$\mathcal{L}\{\cos 7t\}$$
$$\mathcal{L}\{\sinh 3t\}$$

$$\mathcal{L}\{\cos 7t\}$$

starts  
(am)  
Topic

Laplace Transform on Derivative :-

I, Division by "t"

II Multiply by "t"

Q.1

Ex 1

If  $\mathcal{L}\{f(t)\} = f(s)$  then

$$\mathcal{L}\{t f(t)\} = (-1) \frac{d}{ds} f(s)$$

$$\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} f(s)$$

$$\mathcal{L}\{t^3 f(t)\} = (-1)^3 \frac{d^3}{ds^3} f(s) \dots$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$$



Find  $\mathcal{L}\{t \sin at\}$

Let  $f(t) = \sin at$

Take LT on b.s

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2} = f(s)$$

$$\mathcal{L}\{t f(t)\} = (-1) \frac{d}{ds} f(s)$$

$$\mathcal{L}\{t \sin at\} = (-1) \frac{d}{ds} \left[ \frac{a}{(s^2 + a^2)} \right]$$

$$= -a \left[ \frac{-1}{(s^2 + a^2)^2} \cdot 2s \right]$$

$$= \frac{2as}{(s^2 + a^2)^2}$$

$$\frac{d}{dn} \frac{1}{n} = -\frac{1}{n^2}$$

29 Find  $\mathcal{L}\{e^{-4t} t^2\}$

Let  $f(t) = e^{-4t}$

Take LT on b.s

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-4t}\} = \frac{1}{s+4} = f(s)$$

$$\mathcal{L}\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} f(s)$$

$$\mathcal{L}\{t^2 e^{-4t}\} = \frac{d^2}{ds^2} \left( \frac{1}{s+4} \right)$$

$$= \frac{d}{ds} \left[ \frac{d}{ds} \left( \frac{1}{s+4} \right) \right]$$

$$= \frac{d}{ds} \left[ \frac{-1}{(s+4)^2} \right]$$

$$= (-1) \frac{d}{ds} (s+4)^{-2}$$

$$= (-1)(-2)(s+4)^{-3}$$

$$= \frac{2}{(s+4)^3}$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a}}$$



34 Find L.T of  $\int_0^t e^{-2t} \cos 3t \, dt$  (or) Find  $\mathcal{L} \{ t e^{-2t} \cos 3t \}$

let  $f(t) = \cos 3t$

$$\mathcal{L} \{ f(t) \} = \mathcal{L} \{ \cos 3t \} = \frac{s}{s^2 + 9} = f(s)$$

By 1<sup>st</sup> shifting theorem,

$$\mathcal{L} \{ e^{at} f(t) \} = f(s-a)$$

$$\mathcal{L} \{ e^{-2t} \cos 3t \} = \frac{s+2}{(s+2)^2 + 9} = \frac{s+2}{s^2 + 4s + 13}$$

Multiply by  $t$

$$\mathcal{L} \{ t f(t) \} = (-1) \frac{d}{ds} f(s)$$

$$\mathcal{L} \{ t e^{-2t} \cos 3t \} = (-1) \frac{d}{ds} \frac{s+2}{s^2 + 4s + 13}$$

$$= (-1) \left[ \frac{(s^2 + 4s + 13)(1) - (s+2)(2s+4)}{(s^2 + 4s + 13)^2} \right]$$

$$= (-1) \left[ \frac{s^2 + 4s + 13 - 2s^2 - 4s - 4s - 8}{(s^2 + 4s + 13)^2} \right]$$

$$= \frac{s^2 + 4s + 5}{(s^2 + 4s + 13)^2}$$

$$\sin at = \frac{a}{s^2 + a^2}$$

400 Find  $\mathcal{L} \{ t e^{2t} \sin 3t \}$

let  $f(t) = \sin 3t$

$$\mathcal{L} \{ f(t) \} = \mathcal{L} \{ \sin 3t \} = \frac{3}{s^2 + 9} = f(s)$$

By 1<sup>st</sup> shifting theorem

$$\mathcal{L} \{ e^{at} f(t) \} = f(s-a)$$

$$\mathcal{L} \{ e^{2t} \sin 3t \} = \frac{3}{s^2 - 4s + 13}$$

$$= \frac{3}{s^2 - 4s + 13}$$



multiply by 't'

$$\mathcal{L}\{t e^{2t} \sin 3t\} = (-1) \frac{d}{ds} f(s)$$

$$= (-1) \left[ \frac{-1}{(s^2 - 4s + 13)^2} (2s - 4) \right]$$

$$= \frac{6s - 12}{(s^2 - 4s + 13)^2}$$

Q find  $\mathcal{L}\{t \sin^2 t\}$

sol<sup>n</sup> let  $f(t) = \sin^2 t = \frac{1 - \cos 2t}{2}$

$$\mathcal{L}\{f(t)\} = \frac{1}{2} \mathcal{L}\{1 - \cos 2t\} = \frac{1}{2} \left[ \frac{1}{s} - \frac{s}{s^2 + 4} \right] = f(s)$$

$$\mathcal{L}\{t f(t)\} = (-1) \frac{d}{ds} f(s)$$

$$\mathcal{L}\{t \sin^2 t\} = (-1) \frac{d}{ds} \frac{1}{2} \left[ \frac{1}{s} - \frac{s}{s^2 + 4} \right]$$

$$= -\frac{1}{2} \left[ -\frac{1}{s^2} - \frac{(s^2 + 4)(1) - s \cdot 2s}{(s^2 + 4)^2} \right]$$

$$= -\frac{1}{2} \left[ -\frac{1}{s^2} - \frac{[s^2 + 4 - 2s^2]}{(s^2 + 4)^2} \right]$$

$$= -\frac{1}{2} \left[ -\frac{1}{s^2} - \frac{(4 - s^2)}{(s^2 + 4)^2} \right]$$

$$= \frac{1}{2} \left[ \frac{1}{s^2} + \frac{(4 - s^2)}{(s^2 + 4)^2} \right]$$

$$\frac{u}{v} = \frac{vu' - uv'}{v^2}$$



x by 't'

$$\mathcal{L}\{t f(t)\} = (-1) \frac{d}{ds} f(s)$$

$$= (-1) \left[ \frac{d}{ds} \frac{1}{s^2 + 2s + 2} \right]$$

$$= (-1) \left[ \frac{-1}{(s^2 + 2s + 2)^2} (2s + 2) \right]$$

$$= \frac{2s + 2}{(s^2 + 2s + 2)^2}$$

pak (2m) (99)

Division by 't'

Statement: If  $\mathcal{L}\{f(t)\} = f(s)$  then  $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$

Q. Find  $\mathcal{L}\left\{\frac{1-e^t}{t}\right\}$

Soln Let  $f(t) = 1 - e^t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 - e^t\} = \frac{1}{s} - \frac{1}{s-1} = f(s)$$

Division by 't'

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$\mathcal{L}\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left[ \frac{1}{s} - \frac{1}{s-1} \right] ds$$

$$= \left[ \log s - \log(s-1) \right]_s^\infty$$

$$= \left[ \log \frac{s}{s-1} \right]_s^\infty$$

$$= \left[ \log \frac{s}{s(1-\frac{1}{s})} \right]_s^\infty$$

$$= -\log \left[ 1 - \frac{1}{s} \right]$$

$$s = \log \frac{s-1}{s}$$

Whenever (as) is their take out highest power common from N & D



Find  $\mathcal{L}\left\{\frac{\sin \omega t}{t}\right\}$

Let  $f(t) = \sin \omega t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin \omega t\} = \frac{\omega}{s^2 + \omega^2} = f(s)$$

Division by  $t$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$\mathcal{L}\left\{\frac{\sin \omega t}{t}\right\} = \int_s^\infty \frac{\omega}{s^2 + \omega^2} ds$$

$$= \omega \int_s^\infty \frac{1}{s^2 + \omega^2} ds$$

$$= \omega \left[ \frac{1}{\omega} \tan^{-1} \left( \frac{s}{\omega} \right) \right]_s^\infty$$

$$= \tan^{-1} \frac{\infty}{\omega} - \tan^{-1} \frac{s}{\omega}$$

$$= \frac{\pi}{2} - \tan^{-1} \frac{s}{\omega}$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

Find  $\mathcal{L}\left\{\frac{1 - \cos t}{t}\right\}$

Let  $f(t) = 1 - \cos t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1 - \cos t\} = \frac{1}{s} - \frac{s}{s^2 + 1} = f(s)$$

Division by  $t$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$\mathcal{L}\left\{\frac{1 - \cos t}{t}\right\} = \int_s^\infty \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) ds$$

$$= \left[ \log s - \frac{1}{2} \log(s^2 + 1) \right]_s^\infty$$

$$= \left[ \log s - \log(s^2 + 1)^{1/2} \right]_s^\infty$$

$$= \left[ \log \frac{s}{\sqrt{s^2 + 1}} \right]_s^\infty = \log \left( \frac{s}{\sqrt{1 + \frac{1}{s^2}}} \right) = \left[ \log \frac{1}{\sqrt{1 + \frac{1}{s^2}}} - \log \frac{1}{\sqrt{1 + \frac{1}{s^2}}} \right]$$

$$\int \frac{f(x)}{x} dx = \log f(x)$$

$$= \log 1 - \log s$$



Figure 1. The effect of the concentration of the *Agrobacterium* suspension on the transformation efficiency of *Agrobacterium* strains.

12345678910111213141516171819202122232425262728293031323334353637383940414243444546474849505152535455565758596061626364656667686970717273747576777879808182838485868788899091929394959697989910010110210310410510610710810911011111211311411511611711811912012112212312412512612712812913013113213313413513613713813914014114214314414514614714814915015115215315415515615715815916016116216316416516616716816917017117217317417517617717817918018118218318418518618718818919019119219319419519619719819920020120220320420520620720820921021121221321421521621721821922022122222322422522622722822923023123223323423523623723823924024124224324424524624724824925025125225325425525625725825926026126226326426526626726826927027127227327427527627727827928028128228328428528628728828929029129229329429529629729829930030130230330430530630730830931031131231331431531631731831932032132232332432532632732832933033133233333433533633733833934034134234334434534634734834935035135235335435535635735835936036136236336436536636736836937037137237337437537637737837938038138238338438538638738838939039139239339439539639739839940040140240340440540640740840941041141241341441541641741841942042142242342442542642742842943043143243343443543643743843944044144244344444544644744844945045145245345445545645745845946046146246346446546646746846947047147247347447547647747847948048148248348448548648748848949049149249349449549649749849950050150250350450550650750850951051151251351451551651751851952052152252352452552652752852953053153253353453553653753853954054154254354454554654754854955055155255355455555655755855956056156256356456556656756856957057157257357457557657757857958058158258358458558658758858959059159259359459559659759859960060160260360460560660760860961061161261361461561661761861962062162262362462562662762862963063163263363463563663763863964064164264364464564664764864965065165265365465565665765865966066166266366466566666766866967067167267367467567667767867968068168268368468568668768868969069169269369469569669769869970070170270370470570670770870971071171271371471571671771871972072172272372472572672772872973073173273373473573673773873974074174274374474574674774874975075175275375475575675775875976076176276376476576676776876977077177277377477577677777877978078178278378478578678778878979079179279379479579679779879980080180280380480580680780880981081181281381481581681781881982082182282382482582682782882983083183283383483583683783883984084184284384484584684784884985085185285385485585685785885986086186286386486586686786886987087187287387487587687787887988088188288388488588688788888989089189289389489589689789889990090190290390490590690790890991091191291391491591691791891992092192292392492592692792892993093193293393493593693793893994094194294394494594694794894995095195295395495595695795895996096196296396496596696796896997097197297397497597697797897998098198298398498598698798898999099199299399499599699799899910001001100210031004100510061007100810091010101110121013101410151016101710181019102010211022102310241025102610271028102910301031103210331034103510361037103810391040104110421043104410451046104710481049105010511052105310541055105610571058105910601061106210631064106510661067106810691070107110721073107410751076107710781079108010811082108310841085108610871088108910901091109210931094109510961097109810991100110111021103110411051106110711081109111011111112111311141115111611171118111911201121112211231124112511261127112811291130113111321133113411351136113711381139114011411142114311441145114611471148114911501151115211531154115511561157115811591160116111621163116411651166116711681169117011711172117311741175117611771178117911801181118211831184118511861187118811891190119111921193119411951196119711981199120012011202120312041205120612071208120912101211121212131214121512161217121812191220122112221223122412251226122712281229123012311232123312341235123612371238123912401241124212431244124512461247124812491250125112521253125412551256125712581259126012611262126312641265126612671268126912701271127212731274127512761277127812791280128112821283128412851286128712881289129012911292129312941295129612971298129913001

1000

$$2 - \frac{1}{2} = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}$$

A photograph of a handwritten document, possibly a ledger or account book. A prominent diagonal line runs across the page. There are several blue ink markings, including what appear to be numbers and symbols, scattered across the page. The handwriting is in blue ink and is somewhat faded and difficult to read. The document is placed on a light-colored surface.

*[Faint handwritten notes at the bottom of the page]*

*[Faint handwritten notes, possibly bleed-through from the reverse side.]*



Q Find  $\mathcal{L}\{t e^{-2t} \cosh t\}$

Soln Let  $f(t) = \cosh t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cosh t\} = \frac{s}{s^2+1}$$

1st shifting theorem

$$\mathcal{L}\{e^{-at} f(t)\} = f(s-a)$$

$$\mathcal{L}\{e^{-2t} \cosh t\} = \frac{s+2}{(s+2)^2+1} = \frac{s+2}{s^2+4s+3} = f(s)$$

x by 't'

$$\mathcal{L}\{t f(t)\} = (-1) \frac{d}{ds} f(s)$$

$$\mathcal{L}\{t e^{-2t} \cosh t\} = (-1) \frac{d}{ds} \left[ \frac{s+2}{s^2+4s+3} \right]$$

$$= (-1) \left[ \frac{(s^2+4s+3)(1) - (s+2)(2s+4)}{(s^2+4s+3)^2} \right]$$

$$= (-1) \left[ \frac{-s^2+4s+3-2s^2-4s-4s-8}{(s^2+4s+3)^2} \right]$$

$$= \frac{s^2+4s+5}{(s^2+4s+3)^2}$$

Q Find  $\mathcal{L}\{t e^{-t} \sin t\}$

Sol Let  $f(t) = \sin t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

By 1st shifting theorem

$$\mathcal{L}\{e^{at} f(t)\} = f(s-a)$$

$$\mathcal{L}\{e^{-t} \sin t\} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2} = f(s)$$



Imp find  $\int_0^\infty \frac{\sin t}{t} dt$

Soln Let  $f(t) = \sin t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty f(s) ds$$

Refer the below fig

$$\int_s^\infty \frac{1}{s^2+1} ds$$

$$= [\tan^{-1} s]_s^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x$$

Ex 1 find  $\mathcal{L}\left\{\frac{\sin t}{t}\right\}$

Soln Let  $f(t) = \sin t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$$

Division by 't'

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$\mathcal{L}\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1} ds$$

$$= [\tan^{-1} s]_s^\infty$$

$$= \tan^{-1} \infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\int \frac{1}{1+x^2} dx = \tan^{-1} x$$

Imp (99)

Find  $\int_0^\infty \frac{\sin t}{t} dt$

Let  $f(t) = \sin t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1} = f(s)$$



Division by  $t$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$\int_0^\infty \frac{f(t)}{t} dt = \int_0^s f(s) ds + \int_s^\infty f(s) ds$$

$$= \int_0^s \frac{1}{s^2+1} ds + \int_s^\infty \frac{1}{s^2+1} ds$$

$$= [\tan^{-1}s]_0^s + [\tan^{-1}s]_s^\infty$$

$$= [\cancel{\tan^{-1}s} - \cancel{\tan^{-1}0}] + \tan^{-1}\infty - \cancel{\tan^{-1}s}$$

$$= -0 + \pi/2$$

$$\int_0^\infty \frac{\sin t}{t} dt = \pi/2$$

//

Q. Find  $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$

Soln. Let  $f(t) = e^{-t} - e^{-3t}$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{e^{-t} - e^{-3t}\} = \frac{1}{s+1} - \frac{1}{s+3} = f(s)$$

Division by 't',

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt = \int_0^s \left[ \frac{1}{s+1} - \frac{1}{s+3} \right] ds + \int_s^\infty \left( \frac{1}{s+1} - \frac{1}{s+3} \right) ds$$

$$= [\log(s+1) - \log(s+3)]_0^s + [\log(s+1) - \log(s+3)]_s^\infty$$

$$= \left[ \log \frac{(s+1)}{(s+3)} \right]_0^s + \left[ \log \frac{(s+1)}{(s+3)} \right]_s^\infty$$

$$= \left( \log \frac{(s+1)}{(s+3)} \right)_0^s + \log \frac{s/(s+1/s)}{s/(s+1/s)} \rightarrow \infty$$

$$\left[ \frac{1}{\infty} = 0 \right]$$



$$\begin{aligned}
 &= \log \frac{s+1}{s+3} - \log \frac{1}{3} + \log \left( \frac{1+0}{1+0} \right) - \log \left( \frac{1+1/s}{1+3/s} \right) \\
 &= \log \frac{s+1}{s+3} - [\log 1 - \log 3] + \log 1 - \log \left( \frac{s+1}{s+3} \right) \\
 &= -\log 1 + \log 3 + \log 1 \\
 &= \log 3 //
 \end{aligned}$$

$$\therefore \int_0^{\infty} \frac{e^{-t} - e^{-3t}}{t} dt = \log 3 //$$

Ex 19 find  $\mathcal{L}\left\{\frac{e^{-2t} \sin 2t}{t}\right\}$

Ex 20 Let  $f(t) = \sin 2t$   
 $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4} = f(s)$

By 1st sh/ft theorem

$$\mathcal{L}\{e^{-at} f(t)\} = f(s-a)$$

$$\mathcal{L}\{e^{-2t} \sin 2t\} = \frac{2}{(s+2)^2+4} = f(s)$$

Division by  $t$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} f(s) ds$$

$$\mathcal{L}\left\{\frac{e^{-2t} \sin 2t}{t}\right\} = \int_s^{\infty} \frac{2}{(s+2)^2+2^2} ds$$

$$= 2 \int_s^{\infty} \frac{1}{(s+2)^2+2^2} ds$$

$$= 2 \cdot \left[ \frac{1}{2} \tan^{-1} \frac{s+2}{2} \right]_s^{\infty}$$

$$= \tan^{-1} \infty - \tan^{-1} \frac{s+2}{2}$$

$$= \frac{\pi}{2} - \tan^{-1} \frac{s+2}{2} //$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$$



89 find  $\mathcal{L}\left\{\frac{\cos 2t - \cos 3t}{t}\right\}$  / imp

Let  $f(t) = \cos 2t - \cos 3t$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\cos 2t - \cos 3t\} = \frac{s}{s^2+4} - \frac{s}{s^2+9} = f(s)$$

Division by  $t$

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty f(s) ds$$

$$\mathcal{L}\left\{\frac{\cos 2t - \cos 3t}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+4} - \frac{s}{s^2+9}\right) ds$$

$$= \frac{1}{2} \int_s^\infty \left[ \frac{2s}{s^2+4} - \frac{2s}{s^2+9} \right] ds$$

$$\frac{N_r}{D_r} = \frac{\text{up}}{\text{down}}$$

$$= \frac{1}{2} \left[ \log(s^2+4) - \log(s^2+9) \right]_s^\infty$$

$$= \frac{1}{2} \left[ \log \frac{s^2+4}{s^2+9} \right]_s^\infty$$

$$= \frac{1}{2} \left[ \log \frac{s^2(1+\frac{4}{s^2})}{s^2(1+\frac{9}{s^2})} \right]_s^\infty$$

$$= \frac{1}{2} \left[ \log \frac{(1+0)}{(1+0)} - \log \frac{(1+4/s^2)}{(1+9/s^2)} \right]$$

$$= \frac{1}{2} \left[ \log 1 - \log \frac{s^2+4}{s^2+9} \right]$$

$$= \frac{1}{2} \log \frac{s^2+9}{s^2+4}$$

$$\log 1 = 0$$

$$= \log \sqrt{\frac{s^2+9}{s^2+4}}$$



# Inverse Laplace Transform

[5+5+2 (or) 5]

If  $\mathcal{L}\{f(t)\} = f(s)$  then  $\mathcal{L}^{-1}\{f(s)\} = f(t)$

$$\mathcal{L}\{f(t)\} = f(s)$$

$$\bullet \mathcal{L}\{1\} = \frac{1}{s}$$

$$\bullet \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\bullet \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\bullet \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\bullet \mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$$

$$\bullet \mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$$

$$\bullet \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

$$\bullet \mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}$$

$$\bullet \mathcal{L}\{e^{at}f(t)\} = f(s-a)$$

$$\bullet \mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

$$\bullet \mathcal{L}^{-1}\{f(s)\} = f(t)$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \cosh at \sin at$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$$

$$\bullet \mathcal{L}^{-1}\{f(s-a)\} = e^{at}\mathcal{L}^{-1}\{f(s)\} = e^{at}f(t)$$

$$\bullet \mathcal{L}^{-1}\left\{\frac{1}{(s-a)^2}\right\} = e^{at}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = e^{at}t$$

Ex: Find  $\mathcal{L}^{-1}\left\{\frac{2s-5}{s^2-4}\right\}$

Ans:  $2s-5$



$$= 2 \frac{s}{s^2 - 2^2} - \frac{5}{2} \cdot \frac{2}{s^2 - 2^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = 2 \mathcal{L}^{-1}\left\{\frac{s}{s^2 - 2^2}\right\} - \frac{5}{2} \mathcal{L}^{-1}\left\{\frac{2}{s^2 - 2^2}\right\}$$

$$f(t) = 2 \cosh 2t - \frac{5}{2} \sinh 2t$$

9th April 17

Q9 Find  $\mathcal{L}^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

Soln

$$\text{Let } f(s) = \frac{2s+1}{s(s+1)} = \frac{s+s+1}{s(s+1)} = \frac{s}{s(s+1)} + \frac{s+1}{s(s+1)} = \frac{1}{s+1}$$

$$f(s) = \frac{1}{s+1} + \frac{1}{s}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{1}{s}\right\}$$

$$f(t) = e^{-t} + 1$$

(OR)

$$f(s) = \frac{2s+1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$2s+1 = A(s+1) + B(s)$$

$$s=0 \Rightarrow 1 = A$$

$$s=-1 \Rightarrow -1 = -B \quad \therefore B=1$$

$$f(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{1}{s+1}\right\}$$

$$= 1 + e^{-t}$$



39 Find  $\mathcal{L}^{-1} \left\{ \frac{3(s^2-2)^2}{2s^5} \right\}$

Soln Let  $f(s) = \frac{3(s^2-2)^2}{2s^5} = \frac{3}{2} \left( \frac{s^4 - 4s^2 + 4}{s^5} \right)$

$$f(s) = \frac{3}{2} \left[ \frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5} \right]$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{3}{2} \mathcal{L}^{-1} \left\{ \frac{1}{s} - 2 \cdot \frac{2}{s^{2+1}} + \frac{4}{4!} \frac{4!}{s^{4+1}} \right\}$$

$$f(t) = \frac{3}{2} \left( 1 - 2t^2 + \frac{1}{6}t^4 \right)$$

49 Find  $\mathcal{L}^{-1} \left\{ \frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2} \right\}$

Soln Let  $f(s) = \frac{2s-5}{4s^2+25} + \frac{4s-18}{9-s^2}$

$$= \frac{2s}{4s^2+25} - \frac{5}{4s^2+25} - \left[ \frac{4s}{s^2-9} - \frac{18}{s^2-9} \right]$$

$$= \frac{2s}{4(s^2+\frac{25}{4})} - \frac{5}{4(s^2+\frac{25}{4})} - \frac{4s}{s^2-3^2} + 6 \cdot \frac{3}{s^2-3^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+(\frac{5}{2})^2} \right\} - \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{5/2}{s^2+(\frac{5}{2})^2} \right\} - 4 \mathcal{L}^{-1} \left\{ \frac{s}{s^2-3^2} \right\} + 6 \mathcal{L}^{-1} \left\{ \frac{3}{s^2-3^2} \right\}$$

$$f(t) = \frac{1}{2} \cos \frac{5}{2}t - \frac{1}{2} \sin \frac{5}{2}t - 4 \cosh 3t + 6 \sinh 3t$$

59 Find  $\mathcal{L}^{-1} \left\{ \frac{s^2+9s-9}{s^3-9s} \right\}$

Soln



59  $f(s) = \frac{s^2 + 9s - 9}{s(s^2 - 9)} = \frac{A}{s} + \frac{B}{s^2 - 9}$

$$s^2 + 9s - 9 = A(s^2 - 9) + Bs$$

$$s=0 \Rightarrow -9 = -9A \Rightarrow A=1$$

$$s \text{ coeff } 9 = B$$

$$f(s) = \frac{1}{s} + \frac{9}{s^2 - 9}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} + 3 \cdot \frac{3}{s^2 - 3^2}\right\}$$

$$f(t) = 1 + 3 \sinh 3t$$

69 Find  $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2 + \omega^2)}\right\}$

soln Let  $f(s) = \frac{1}{s^2(s^2 + \omega^2)}$

$$f(s) = \frac{1}{\omega^2} \left[ \frac{1}{s^2} - \frac{1}{s^2 + \omega^2} \right]$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{\omega^2} \mathcal{L}^{-1}\left\{ \frac{1}{s^2} - \frac{1}{s^2 + \omega^2} \right\}$$

$$f(t) = \frac{1}{\omega^2} \left[ t - \frac{1}{\omega} \sin \omega t \right]$$

OR  $f(s) = \frac{1}{s^2(s^2 + \omega^2)} = \frac{A}{s^2} + \frac{B}{s^2 + \omega^2}$

$$1 = A(s^2 + \omega^2) + Bs^2$$

Compare const

$$1 = A\omega^2 \Rightarrow A = \frac{1}{\omega^2}$$

1.

$$(5) 4 3$$

$$= 4 - 3$$

$$= 1$$

add 2nd height

$$= 4 + 1$$

$$= 5$$

$$= \frac{1}{\omega^2} \frac{s^2 + \omega^2 + s^2}{s^2(s^2 + \omega^2)}$$

$$\Rightarrow \frac{1}{s^2(s^2 + \omega^2)}$$



$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{\omega} \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{\omega} \frac{\omega}{s^2 + \omega^2}\right\}$$

$$f(t) = \frac{1}{\omega} \left[ t - \frac{1}{\omega} \sin \omega t \right]$$

Ex Find  $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$

soln Mthd : 1

$$\text{Let } f(s) = \frac{1}{(s-1)(s+2)}$$

$$= \frac{1}{3} \left[ \frac{1}{s-1} - \frac{1}{s+2} \right] \quad (\text{OR})$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+2}\right\}$$

$$f(t) = \frac{1}{3} [e^t - e^{-2t}]$$

Mthd : 2

$$f(s) = \frac{1}{(s-1)(s+2)} = \frac{A}{s-1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s-1)$$

$$s=1 \Rightarrow 1 = 3A \Rightarrow A = \frac{1}{3}$$

$$s=2 \Rightarrow 1 = -3B \Rightarrow B = -\frac{1}{3}$$

$$f(s) = \frac{1}{3(s-1)} - \frac{1}{3(s+2)}$$

$$= \frac{1}{3} \left[ \frac{1}{s-1} - \frac{1}{s+2} \right]$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{3} \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{s+2}\right\}$$

$$f(t) = \frac{1}{3} [e^t - e^{-2t}]$$

Ex Find  $\mathcal{L}^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

sol Let  $f(s) = \frac{3s-8}{4s^2+25} = \frac{3s}{4s^2+25} - \frac{8}{4s^2+25}$

$$4/s^2$$

$$f(s) = \frac{3s}{4(s^2 + \frac{25}{4})} - \frac{8}{4(s^2 + \frac{25}{4})}$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{3}{4} \mathcal{L}^{-1}\left\{\frac{s}{s^2 + (\frac{5}{2})^2}\right\} - \frac{2}{\frac{5}{2}} \mathcal{L}^{-1}\left\{\frac{5/2}{s^2 + (\frac{5}{2})^2}\right\}$$

$$f(t) = \frac{3}{4} \cos \frac{5}{2} t - \frac{4}{5} \sin \frac{5}{2} t$$

Ex Find  $\mathcal{L}^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$

sol  $f(s) = \frac{s^2-3s+4}{s^3} = \frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}$



$$\frac{1}{2} = \frac{1}{2} - \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2} - \frac{1}{2}$$

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$$\frac{1}{2} = \frac{1}{2} - \frac{1}{2}$$

$$\frac{1}{2} = \frac{1}{2} - \frac{1}{2}$$







Answer: Suppose  $f(x) = \frac{1}{x^2}$

we are given  $f(x) = \frac{1}{x^2}$  then apply the

power rule

power rule

$$f(x) = \frac{1}{x^2} = x^{-2} \quad \frac{d}{dx} x^{-2} = \frac{-2x^{-3}}{1} = \frac{-2}{x^3}$$

$$f'(x) = -\frac{2}{x^3} = -2x^{-3}$$

$$x=1 \Rightarrow f' = -2 \Rightarrow f' = -\frac{2}{1^3}$$

$$x=2 \Rightarrow f' = -\frac{2}{2^3} = -\frac{1}{4}$$

$$f'(x) = -\frac{2}{x^3} = -\frac{2}{(x^3)}$$

$$f'(x) = -\frac{2}{x^3} = -2x^{-3}$$

Answer:  $f(x) = \frac{1}{x^2}$

$$f(x) = \frac{1}{x^2}$$

$$(x^{-2})$$

$$f'(x) = -2x^{-3}$$

$$f'(x) = -\frac{2}{x^3}$$

Answer

$$f'(x) = -\frac{2}{x^3}$$

Key



1.  $\frac{1}{x^2} = x^{-2}$

2.  $\frac{d}{dx} x^{-2} = -2x^{-3}$

3.  $= -2x^{-3}$

4.  $= -\frac{2}{x^3}$

5.  $= -\frac{2}{x^3}$

$$\frac{d}{dx} \left( \frac{1}{x^2} \right) = -\frac{2}{x^3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$\frac{d}{dx} \left( \frac{1}{x^2} \right) = -\frac{2}{x^3}$$

$$\frac{d}{dx} x^{-2} = -2x^{-3}$$

$$= -\frac{2}{x^3}$$

$$= -\frac{2}{x^3}$$

$$= -\frac{2}{x^3}$$

$$= -\frac{2}{x^3}$$

$$= -\frac{2}{x^3}$$

$$= -\frac{2}{x^3}$$



$$f(s) = \frac{1}{s-1} + \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{1}{s+2}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s-1)^3} \right\}$$

$$\mathcal{L}\{f(s)\} = \mathcal{L} \left\{ \frac{1}{s-1} + \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{1}{s+2} \right\} = e^t \mathcal{L}^{-1} \left\{ \frac{1}{2} \right\}$$

$$= e^t + t e^t - \frac{1}{2} t^2 e^t + e^{-2t} = \frac{1}{2} e^t t^2$$

VR 1000

Find  $\mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2+a^2)(s^2+b^2)} \right\}$

$$f(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}$$

$$s^2 = A(s^2+b^2) + B(s^2+a^2)$$

Constant  $\Rightarrow 0 = Ab^2 + Ba^2 \rightarrow \textcircled{1}$

$s^2$  coeff  $\rightarrow 1 = A + B \rightarrow \textcircled{2}$  Solve

$$f(s) = \left[ \frac{a^2}{(a^2-b^2)(s^2+a^2)} - \frac{b^2}{(a^2-b^2)(s^2+b^2)} \right]$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{a^2-b^2} \mathcal{L}^{-1} \left[ a \cdot \frac{a}{s^2+a^2} - b \cdot \frac{b}{s^2+b^2} \right]$$

$$= \frac{1}{a^2-b^2} [a \sin at - b \sin bt]$$

$$0 = Ab^2 + Ba^2$$



$$f(s) = \frac{1}{s-1} + \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{1}{s+2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^3}\right\} = e^t \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}$$

$$\mathcal{L}\{f(s)\} = \mathcal{L}\left\{\frac{1}{s-1} + \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{1}{s+2}\right\} = e^t \mathcal{L}^{-1}\left\{\frac{1}{s} + \frac{2}{s^2} - \frac{1}{s^3}\right\}$$

$$= e^t + t e^t - \frac{1}{2} t^2 e^t + e^{-2t} = \frac{1}{2} e^t t^2$$

Worked Example

Find  $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

$$+ \frac{2}{s^2} = \frac{2}{s^2}$$

$$f(s) = \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}$$

$$s^2 = A(s^2+b^2) + B(s^2+a^2)$$

Constant  $\Rightarrow 0 = Ab^2 + Ba^2 \rightarrow \textcircled{1}$

$s^2$  coeff  $\rightarrow 1 = A + B \rightarrow \textcircled{2}$  Solve  $\textcircled{1}$  &  $\textcircled{2}$

$$f(s) = \left[ \frac{a^2}{(a^2-b^2)(s^2+a^2)} - \frac{b^2}{(a^2-b^2)(s^2+b^2)} \right]$$

$$0 = Ab^2 + Ba^2$$

$$b^2 = -Ab^2 + Bb^2$$

$$-b^2 = B(a^2-b^2)$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{a^2-b^2} \mathcal{L}^{-1}\left[ a \cdot \frac{a}{s^2+a^2} - b \cdot \frac{b}{s^2+b^2} \right]$$

$$\frac{-b^2}{a^2-b^2} = B \quad \text{sub (1)}$$

$$= \frac{1}{a^2-b^2} [a \cdot \sin at - b \sin bt]$$

$$0 = Ab^2 + \left(\frac{b^2}{a^2-b^2}\right)a^2$$

$$f(t) = \frac{a \sin at - b \sin bt}{a^2-b^2}$$

$$0 = A - \frac{a^2}{a^2-b^2}$$



Q Find  $\mathcal{L}^{-1} \left\{ \frac{s+2}{s^2-6s+8} \right\}$

Soln Let  $f(s) = \frac{s+2}{s^2-6s+8} = \frac{s+2}{(s-2)(s-4)} = \frac{A}{s-2} + \frac{B}{s-4}$

$$s+2 = A(s-4) + B(s-2)$$

$$s=2 \Rightarrow 4 = -2A \Rightarrow A = -2$$

$$s=4 \Rightarrow 6 = 2B \Rightarrow B = 3$$

$$f(s) = \frac{-2}{s-2} + \frac{3}{s-4}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{-2}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s-4}\right\}$$

$$f(t) = -2e^{2t} + 3e^{4t}$$

Q Find  $\mathcal{L}^{-1} \left\{ \frac{3s+1}{(s-1)(s^2+1)} \right\}$

Soln  $f(s) = \frac{3s+1}{(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+1}$

$$3s+1 = A(s^2+1) + (Bs+C)(s-1)$$

$$s=1 \Rightarrow 4 = 2A \Rightarrow A=2$$

Compare  $s^2$  coeff  $\Rightarrow 0 = A+B \Rightarrow B = -A = -2$

Const  $1 = A-C \Rightarrow C = A-1 = 2-1 = 1$

$$f(s) = \frac{2}{s-1} + \frac{-2s+1}{s^2+1}$$



Find  $\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)(s^2+9)} \right\}$

soln  $f(s) = \frac{s}{(s^2+4)(s^2+9)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+9}$

$$s = (As+B)(s^2+9) + (Cs+D)(s^2+4)$$

Constant  $0 = 9B+4D \rightarrow (1)$

$s^3$  coeff  $0 = A+C \rightarrow (2)$

$s$  coeff  $1 = 9A+4C \rightarrow (3)$

$$C = -A = -\frac{1}{5}$$

$s^2$  coeff  $0 = B+D \rightarrow (4)$

Solve (2) & (3)

$$(2) \times 4 = 0 = 4A + 4C$$

$$1 = 9A + 4C$$

$$-1 = -5A$$

$$A = +1/5$$

$$f(s) = \frac{s}{5(s^2+2^2)} - \frac{s}{5(s^2+3^2)}$$

Solve (1) & (4)

$$(4) \times 4 \Rightarrow 4B + 4D = 0$$

$$9B + 4D = 0$$

$$-5B = 0$$

$$B = 0, D = 0$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+2^2} \right\} - \frac{1}{5} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+3^2} \right\}$$

$$f(t) = \frac{1}{5} \cos 2t - \frac{1}{5} \cos 3t$$

Find  $\mathcal{L}^{-1} \left\{ \frac{s}{(s+2)^2} \right\}$

$$f(s) = \frac{s}{(s+2)^2} = \frac{(s+2)-2}{(s+2)^2} = \frac{s+2}{(s+2)^2} - \frac{2}{(s+2)^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1} \left\{ \frac{1}{s+2} \right\} - 2 \mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2} \right\}$$

$$f(t) = e^{-2t} - 2te^{-2t}$$



$$\left\{ \frac{s}{2s^2-1} \right\}$$

sol

$$f(s) = \frac{s}{2s^2-1} = \frac{s}{2\left[s^2 - \frac{1}{2}\right]}$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{2} \mathcal{L}^{-1}\left\{ \frac{s}{s^2 - \sqrt{\frac{1}{2}}^2} \right\}$$

$$f(t) = \frac{1}{2} \cosh \frac{1}{\sqrt{2}} t$$

Q

Find  $\mathcal{L}^{-1}\left\{ \frac{s}{(s-4)^5} \right\}$

soln

$$f(s) = \frac{s}{(s-4)^5} = \frac{(s-4)+4}{(s-4)^5} = \frac{s-4}{(s-4)^5} + \frac{4}{(s-4)^5}$$

$$f(s) = \frac{1}{(s-4)^4} + \frac{4}{(s-4)^5}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s-4)^4} + \frac{4}{(s-4)^5} \right\}$$

$$f(t) = e^{4t} \mathcal{L}^{-1}\left\{ \frac{1}{s^4} + \frac{4}{s^5} \right\}$$

$$= e^{4t} \mathcal{L}^{-1}\left\{ \frac{1}{3!} \frac{3!}{s^{3+1}} + \frac{4}{4!} \frac{4!}{s^{4+1}} \right\}$$

$$= e^{4t} \left[ \frac{1}{6} t^3 + \frac{1}{6} t^4 \right]$$

$$\mathcal{L}^{-1}\left\{ \frac{s^n}{s^{n+1}} \right\} = t^n$$

Find  $\mathcal{L}^{-1}\left\{ \frac{1}{s(s+2)^2} \right\}$

Let  $f(s) = \frac{1}{s(s+2)^2} = \frac{A}{s} + \frac{B}{(s+2)} + \frac{C}{(s+2)^2}$



$$1 = A(s+2)^2 + B(s+2) + C$$

$$s=0 \Rightarrow 1 = 4A \Rightarrow A = \frac{1}{4}$$

$$s^2 \text{ coeff } 0 = A + B \Rightarrow B = -A = -\frac{1}{4}$$

$$s \text{ coeff } 0 = 2A + C \Rightarrow C = -2A = -\frac{1}{2}$$

$$f(s) = \frac{1}{4s} - \frac{1}{4(s+2)} - \frac{1}{2(s+2)^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{4} \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= \frac{1}{4}(1) - \frac{1}{4}e^{-2t} - \frac{1}{2}te^{-2t}$$

9 find  $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$

soln let  $f(s) = \frac{1}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$

$$1 = A(s^2+2s+2) + (Bs+C)s$$

$$s=0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$$

$$s^2 \text{ coeff } 0 = A + B \Rightarrow B = -A = -\frac{1}{2}$$

$$s \text{ coeff } 0 = 2A + C \Rightarrow C = -2A \Rightarrow C = -2\left(\frac{1}{2}\right) = -1$$

$$f(s) = \frac{1}{2s} + \frac{-\frac{1}{2}s - 1}{s^2+2s+1+1}$$

$$f(s) = \frac{1}{2s} - \frac{1}{2} \frac{s}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\{f(s)\} = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{s}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$



$$f(t) = \frac{1}{2}(1) - \frac{1}{2}e^{-t} \mathcal{L}^{-1} \left\{ \frac{s}{s^2+1} - \frac{1}{s+1} \right\} = \mathcal{L}^{-1} \left\{ \frac{1}{(s+1)^2+1} \right\}$$

$$= \frac{1}{2} - \frac{1}{2}e^{-t} [\cos t - \sin t] - e^{-t} \sin t$$

$$= \frac{1}{2} - \frac{1}{2}e^{-t} \cos t - \frac{1}{2}e^{-t} \sin t$$

$$+ \frac{1}{2}e^{-t} \sin t - e^{-t} \sin t$$

3 April 17

Convolution theorem: (5m) ~~10~~ (10) Compulsory

Statement: If  $f(t)$  &  $g(t)$  are two functions of 't' such that their Laplace transform is given by

$$\mathcal{L}\{f(t)\} = f(s) \text{ and } \mathcal{L}\{g(t)\} = g(s) \text{ then}$$

$$\mathcal{L}^{-1}\{f(s) \cdot g(s)\} = \int_0^t f(u) \cdot g(t-u) du$$

Q Find  $\mathcal{L}^{-1} \left\{ \frac{1}{(s+1)(s+2)} \right\}$  by using Convolution theorem.

$$f(s) = \frac{1}{s+2} ; g(s) = \frac{1}{s+1}$$

Take  $\frac{1}{s+2}$  as  $f(s)$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} ; \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$f(t) = e^{-2t} ; g(t) = e^{-t}$$

$$e^{at} \frac{1}{s-a} = \frac{1}{s-a}$$

By convolution theorem

$$\mathcal{L}^{-1}\{f(s) \cdot g(s)\} = \int_0^t f(u) \cdot g(t-u) du$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s+2)(s+1)} \right\} = \int_0^t e^{-2u} e^{-(t-u)} du$$

$$= e^{-t} \int_0^t e^{-3u} du$$

29 Find  $\mathcal{L}^{-1}$

soln  $f(s)$

$$\mathcal{L}^{-1}\{f(s)\}$$

$$f(t)$$

By  $\mathcal{L}^{-1}$

$$\mathcal{L}^{-1}$$

$$\int \sin x dx = -\cos x + C$$



$$= e^t \left[ \frac{e^{-3u}}{-3} \right]_0^t$$

$$= \frac{e^t}{-3} [e^{-3t} - e^0]$$

$$= -\frac{1}{3} [e^{-3t} - e^t]$$

$$= \frac{1}{3} [e^t - e^{-3t}] //$$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$-\frac{1}{3} [e^{-3t} - e^t]$$

29 Find  $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2 - a^2)} \right\}$  by using convolution theorem

soln  $f(s) = \frac{1}{s^2 - a^2}$  ;  $g(s) = \frac{1}{s}$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{s^2 - a^2} \right\} ; \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s^2 - a^2} \right\}$$

$$f(t) = \frac{1}{a} \sinh at$$

$$g(t) = 1$$

By convolution theorem

$$\mathcal{L}^{-1}\{f(s) \cdot g(s)\} = \int_0^t f(u) \cdot g(t-u) du$$

$$\mathcal{L}^{-1}\left\{ \frac{1}{s^2 - a^2} \cdot \frac{1}{s} \right\} = \int_0^t \frac{1}{a} \sinh au \cdot \underline{g(t-u)} du$$

$$= \frac{1}{a} \left[ \frac{\cosh au}{a} \right]_0^t$$

$$= \frac{1}{a^2} [ \cosh at - \cosh 0 ]$$

$$= \frac{1}{a^2} [\cosh at - 1] //$$

$$\int \sin x dx = -\cos x$$

$$\int \sinh x dx = \cosh x$$



39 Find  $\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+a^2)^2} \right\}$  by using Convolution theorem

$$f(s) = \frac{s}{s^2+a^2} \quad ; \quad g(s) = \frac{1}{s^2+a^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}; \quad \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+a^2}\right\}$$

$$f(t) = \cos at \quad ; \quad g(t) = \frac{1}{a} \sin at$$

By convolution theorem:

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t f(u)g(t-u)du$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \int_0^t \cos au \cdot \frac{1}{a} \sin a(t-u) du$$

$$= \frac{1}{2a} \int_0^t 2 \cos au \cdot \sin (at-au) du$$

$$\left. \begin{aligned} \cos A \sin B \\ = \frac{1}{2} [\sin(A+B) - \sin(A-B)] \end{aligned} \right\} = \frac{1}{2a} \int_0^t [\sin(au+at-au) - \sin(au-at+au)] du$$

$$\cos(-\theta) = +\cos\theta$$

$$= \frac{1}{2a} \int_0^t [\sin at - \sin(2au-at)] du$$

$$= \frac{1}{2a} \left[ \sin at \cdot u + \frac{\cos(2au-at)}{2a} \right]_0^t$$

$$= \frac{1}{2a} \left[ \sin at \cdot t + \frac{\cos(2at-at)}{2a} - \left( 0 + \frac{\cos(2a \times 0 - at)}{2a} \right) \right]$$

$$= \frac{1}{2a} \left[ t \cdot \sin at + \frac{1}{2a} \cos at - \frac{1}{2a} \cos at \right]$$

$$= \frac{1}{2a} t \sin at //$$



40 Find  $\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$  by convolution theorem

soln let  $f(s) = \frac{s}{s^2+a^2}$  ;  $g(s) = \frac{s}{s^2+b^2}$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\} ; \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\}$$

$$f(t) = \cos at \quad ; \quad g(t) = \cos bt$$

By convolution theorem

$$\mathcal{L}^{-1}\{f(s) \cdot g(s)\} = \int_0^t f(u) \cdot g(t-u) du$$

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos(b(t-u)) du$$

$$= \int_0^t \cos au \cdot \cos(bt-bu) du$$

$$= \frac{1}{2} \int_0^t [2 \cos au \cdot \cos(bt-bu)] du$$

$$= \frac{1}{2} \int_0^t [\cos(au-bu+bt) + \cos(au+bu-bt)] du$$

$$= \frac{1}{2} \left[ \frac{\sin(au-bu+bt)}{a-b} + \frac{\sin(au+bu-bt)}{a+b} \right]_0^t$$

$$= \frac{1}{2} \left[ \left( \frac{\sin(at-bt+bt)}{a-b} + \frac{\sin(at+bt-bt)}{a+b} \right) - \left( \frac{\sin(a \cdot 0 - b \cdot 0 + bt)}{a-b} + \frac{\sin(a \cdot 0 + b \cdot 0 - bt)}{a+b} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{\sin(at)}{a-b} + \frac{\sin at}{a+b} \right] - \frac{1}{2} \left[ \frac{\sin bt}{a-b} + \frac{\sin(-bt)}{a+b} \right]$$

$$= \frac{1}{2} \sin at \left[ \frac{1}{a-b} + \frac{1}{a+b} \right] - \frac{1}{2} \sin bt \left[ \frac{1}{a-b} - \frac{1}{a+b} \right]$$

$$= \frac{\sin at}{2} \left[ \frac{a+b+a-b}{a^2-b^2} \right] - \frac{1}{2} \sin bt \left[ \frac{a+b-a+b}{a^2-b^2} \right]$$



$$= \frac{\sin at \cdot 2a}{2(a^2 - b^2)} - \frac{\sin bt \cdot 2b}{2(a^2 - b^2)}$$

$$= \frac{a \sin at}{a^2 - b^2} - \frac{b \sin bt}{a^2 - b^2}$$

$$\mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2 + a^2)(s^2 + b^2)} \right\} = \frac{1}{a^2 - b^2} \left[ a \sin at - b \sin bt \right]$$

Convolution  
statement  
(v. imp  
in exam)

15 April 12

5Q Using Convolution theorem  $\mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2 + 4)(s^2 + 9)} \right\}$

Soln let  $f(s) = \frac{s^2}{s^2 + 9}$  ;  $g(s) = \frac{s}{s^2 + 4}$

$$\mathcal{L}^{-1} \{ f(s) \} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 9} \right\} ; \mathcal{L}^{-1} \{ g(s) \} = \mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\}$$

$$f(t) = \cos(3t)$$

$$g(t) = \cos 2t$$

By using convolution theorem

$$\mathcal{L}^{-1} \{ f(s) \cdot g(s) \} = \int_0^t f(u) g(t-u) du$$

$$\mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2 + 4)(s^2 + 9)} \right\} = \int_0^t \cos 3u \cos 2(t-u) du$$

$$= \frac{1}{2} \int_0^t \cos 3u \cdot \cos 2(2t - 2u) du$$

$$= \frac{1}{2} \int_0^t [\cos(3u + 2t - 2u) + \cos(3u - 2t + 2u)] du$$

$$= \frac{1}{2} \int_0^t [\cos(u + 2t) + \cos(5u - 2t)] du$$

$$= \frac{1}{2} \left[ \frac{\sin(u + 2t)}{1} + \frac{\sin(5u - 2t)}{5} \right]_0^t$$

$$= \frac{1}{2}$$

$$\cos ax = \frac{\sin ax}{a}$$



$$= \frac{1}{2} \left[ \sin(t+2t) + \sin\left(\frac{5t-2t}{5}\right) - \sin 2t - \frac{\sin(-2t)}{5} \right]$$

$$= \frac{1}{2} \left[ \sin 3t + \frac{\sin 3t}{5} - \sin 2t + \frac{\sin 2t}{5} \right]$$

$$= \frac{1}{2} \left[ \frac{6\sin 3t}{5} - 4\frac{\sin 2t}{5} \right]$$

$$= \frac{1}{2} \cdot \frac{2}{5} [3\sin 3t - 2\sin 2t]$$

$$= \frac{1}{5} [3\sin 3t - 2\sin 2t]$$

$$\mathcal{L}^{-1} \left\{ \frac{s^2}{(s^2+4)(s^2+9)} \right\} = \frac{1}{5} [3\sin 3t - 2\sin 2t]$$

Ex Find  $\mathcal{L}^{-1} \left\{ \frac{1}{s(s^2+1)} \right\}$  by using convolution theorem

Sol  $f(s) = \frac{1}{s^2+1}$        $g(s) = \frac{1}{s}$

$$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} \quad \mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}$$

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$$f(t) = \sin t \quad g(t) = 1$$

By using convolution theorem

$$\mathcal{L}^{-1}\{f(s) \cdot g(s)\} = \int_0^t f(u) \cdot g(t-u) du \quad ) \text{ on } 45 = \text{conv}$$

$$= \int_0^t \sin u \cdot 1 du$$

$$= \left[ -\frac{\cos u}{1} \right]_0^t = -[\cos t + \cos 0]$$



$$\underline{Q.0} \quad \mathcal{L}^{-1} \left\{ \frac{1}{(s-2)(s+3)} \right\}$$

$$\underline{Sol} \quad f(s) = \frac{1}{s+3} \quad ; \quad g(s) = \frac{1}{(s-2)}$$

$$\mathcal{L}^{-1}\{f(s)\}y = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}y \quad ; \quad \mathcal{L}^{-1}\{g(s)\}y = \mathcal{L}^{-1}\left\{\frac{1}{(s-2)}\right\}y$$

$$f(t) = e^{-3t}$$

$$g(t) = e^{2t}$$

By using convolution theorem

$$\mathcal{L}^{-1}\{f(s)g(s)\}y = \int_0^t f(u)g(t-u)du$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-2)(s+3)}\right\}y = \int_0^t e^{-3u} \cdot e^{2(t-u)} du$$

$$= \int_0^t e^{3u} e^{(2t-2u)} du$$

$$= \int_0^t e^{-3u+2t-2u} du$$

$$= \int_0^t e^{-5u+2t} du$$

$$= e^{2t} \left[ \frac{e^{-5u}}{-5} \right]_0^t$$

$$= \frac{e^{2t}}{-5} [e^{-5t} - e^{-5(0)}]$$

$$= \frac{e^{2t}}{-5} [e^{-5t} - 1]$$

$$= \frac{1}{5} [e^{3t} - e^{2t}]$$

$$= \frac{1}{5} [e^{2t} - e^{-3t}] //$$

$$\int e^{-ax} = \frac{e^{-ax}}{-a}$$



89  $\mathcal{L}^{-1} \left\{ \frac{1}{s^2(s^2+1)} \right\}$

90  $\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)(s^2+9)} \right\}$

sol<sup>n</sup>  $f(s) = \frac{1}{(s^2+1)}$  ;  $g(s) = \frac{1}{s^2} = \frac{1}{s^2}$

$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$  ;  $\mathcal{L}^{-1}\{g(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$

$f(t) = \sin t$

$g(t) = t$

By using convolution theorem

$\mathcal{L}^{-1}\{f(s) \cdot g(s)\} = \int_0^t f(u) g(t-u) du$

$= \int_0^t [\sin u \cdot (t-u)] du$

$\sin 0 = 0$   
 $\cos 0 = 1$

$= \int_0^t [t \sin u - \frac{u \sin u}{\frac{u}{v}}] du$

$\int uv = u \cdot v - u'v''$

$= [t(-\cos u) - [u(-\cos u) - (1)(-\sin u)]] du$

$= [-t \cos u + u \cos u + \sin u]_0^t$

$= [-t \cos t + t \cos t + \sin t] - [-t \cos 0 + 0 \cos 0 - \sin 0]$

$= -\sin t + t$

$= t - \sin t$

$\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)(s^2+9)} \right\}$

Imp

90  $f(s) = \frac{s}{s^2+9}$

$g(s) = \frac{1}{s^2+4}$

$\mathcal{L}^{-1}\{f(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\}$

$\mathcal{L}^{-1}\{g(s)\} = \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}$



$$f(t) = \cos 3t$$

$$g(t) = \frac{1}{2} \sin 2t$$

By V.C.T

$$\mathcal{L}\{f(t) \cdot g(t)\} = \int_0^t f(u) g(t-u) du$$

$$\mathcal{L}\left\{\frac{s}{(s^2+9)(s^2+4)}\right\} = \frac{1}{2} \int_0^t \cos 3u \sin 2(t-u) du$$

$$= \frac{1}{2} \int_0^t \cos 3u \sin(2t-2u) du$$

$$= \frac{1}{4} \int_0^t 2 \cos 3u \sin(2t-2u) du$$

$$= \frac{1}{4} \int_0^t [\sin(3u+2t-2u) - \sin(3u-2t+2u)] du$$

$$= \frac{1}{4} \int_0^t [\sin(3u+2t) - \sin(5u-2t)] du$$

$$= \frac{1}{4} \left[ -\cos(u+2t) + \frac{\cos(5u-2t)}{5} \right]_0^t$$

$$= \frac{1}{4} \left[ -\cos(t+2t) + \frac{\cos(5t-2t)}{5} \right]$$

$$- \left[ -\cos(0+2t) + \frac{\cos(0-2t)}{5} \right]$$

$$\mathcal{L}\left\{\frac{1}{(s^2+9)^2}\right\}$$

$$= \frac{1}{4} \left[ -\cos 3t + \frac{\cos 3t}{5} + \cos 2t - \frac{\cos 2t}{5} \right]$$

$$= \frac{1}{4} \left[ \frac{4\cos 2t}{5} - \frac{4\cos 3t}{5} \right]$$

$$= \frac{1}{5} [\cos 2t - \cos 3t]$$



## \* Application LT to D. eqn

NOTE (1)  $\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$

2:  $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$

3:  $\mathcal{L}\{y'''\} = s^3\mathcal{L}\{y\} - s^2y(0) - sy'(0) - y''(0)$

4:  $\mathcal{L}\{y''''\} = s^4\mathcal{L}\{y\} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)$

Ex Solve  $[D^2+1]y = 6\cos 2t$ , given  $y(0)=3$ ,  $y'(0)=1$

Soln Given  $[D^2+1]y = 6\cos 2t$

$$y'' + y = 6\cos 2t$$

Take L.T. on b.s

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = 6\mathcal{L}\{\cos 2t\}$$

$$[s^2\mathcal{L}\{y\} - sy(0) - y'(0)] + \mathcal{L}\{y\} = 6 \frac{s}{s^2+4}$$

$$s^2\mathcal{L}\{y\} - 3s - 1 + \mathcal{L}\{y\} = \frac{6s}{s^2+4}$$

b.s [

$$\mathcal{L}\{y\} [s^2+1] = \frac{6s}{s^2+4} + 3s + 1$$

$$\mathcal{L}\{y\} = \frac{6s}{(s^2+4)(s^2+1)} + \frac{3s+1}{s^2+1}$$

$$y = \mathcal{L}^{-1} \left\{ \frac{6s}{(s^2+4)(s^2+1)} + \frac{3s}{s^2+1} + \frac{1}{s^2+1} \right\}$$

$$= 6\mathcal{L}^{-1} \left\{ \frac{s}{(s^2+4)(s^2+1)} \right\} + 3\cos t + \sin t \rightarrow \textcircled{1}$$



$$f(s) = \frac{s}{s^2+4} ; g(s) = \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$f(t) = \cos 2t ; g(t) = \sin t$$

By using convolution theorem

$$\mathcal{L}^{-1}\{f(s)g(s)\} = \int_0^t f(u)g(t-u) du$$

$$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)(s^2+1)}\right\} = \frac{1}{2} \int_0^t \cos 2u \sin(t-u) du$$

$$= \frac{1}{2} \int_0^t [\sin(2u+t-u) - \sin(2u-t+u)] du$$

$$= \frac{1}{2} \int_0^t [\sin(u+t) - \sin(3u-t)] du$$

$$= \frac{1}{2} \left[ -\cos(u+t) + \frac{\cos(3u-t)}{3} \right]_0^t$$

$$= \frac{1}{2} \left[ -\cos(t+t) + \frac{\cos(3t-t)}{3} + \cos(t) - \frac{\cos(-t)}{3} \right]$$

$$= \frac{1}{2} \left[ -\cos 2t + \frac{\cos 2t}{3} + \cos t - \frac{\cos t}{3} \right]$$

$$= \frac{1}{2} \left[ \frac{-2\cos 2t}{3} + \frac{2\cos t}{3} \right] \text{ sub (1) (1)}$$

$$y = \frac{1}{2} \times \frac{2}{3} [\cos t - \cos 2t] + 3\cos t + \sin t$$

$$= 2\cos t - 2\cos 2t + 3\cos t + \sin t$$

$$= 5\cos t - 2\cos 2t + \sin t$$



① ~~+~~ d

① using LT Sol. D.E.s

$$y'' - 6y' + 9y = 0, \quad y(0) = 2, \quad y'(0) = 9$$

$$② (p^2 + 4p + 3)y = e^t, \quad y = y' = 1.$$

$$③ y'' + y = 2e^t, \quad y(0) = 0, \quad y'(0) = 2$$

$$④ \frac{d^2 y}{dt^2} + 2 \frac{dy}{dt} - 3y = \sin t, \quad y = y' = 0.$$



Solve  $\frac{d^2 y}{dx^2} + y = t \cos 2t$ , given  $y = dy = 0$  at  $t = 0$

Given  $\frac{d^2 y}{dx^2} + y = t \cos 2t$

$$y'' + y = t \cos 2t$$

Take L.T.O. b.s

$$\mathcal{L}\{y''\} + \mathcal{L}\{y\} = \mathcal{L}\{t \cos 2t\}$$

$$s^2 \mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \frac{s^2 - 4}{(s^2 + 4)^2}$$

$$s^2 \mathcal{L}\{y\} + \mathcal{L}\{y\} = \frac{s^2 - 4}{(s^2 + 4)^2}$$

$$\mathcal{L}\{y\} (s^2 + 1) = \frac{s^2 - 4}{(s^2 + 4)^2}$$

$$\mathcal{L}\{y\} = \frac{s^2 - 4}{(s^2 + 4)^2 (s^2 + 1)}$$

$$y = \mathcal{L}^{-1} \left\{ \frac{s^2 - 4}{(s^2 + 4)^2 (s^2 + 1)} \right\}$$

$$\frac{s^2 - 4}{(s^2 + 4)^2 (s^2 + 1)} = \frac{A}{s^2 + 4} + \frac{B}{(s^2 + 4)^2} + \frac{C}{s^2 + 1}$$

$$s^2 - 4 = A(s^2 + 4)(s^2 + 1) + B(s^2 + 1) + C(s^2 + 4)^2$$

$$s^2 = -1, \quad -5 = 9C \Rightarrow C = -\frac{5}{9}$$

$$s^2 = -4, \quad -8 = -3B \Rightarrow B = \frac{8}{3}$$

$$s^2 \text{ coeff} \Rightarrow 0 = A + C \Rightarrow A = -C = \frac{5}{9}$$

$$y = \mathcal{L}^{-1} \left\{ \frac{5}{9} \cdot \frac{2}{s^2 + 2^2} - \frac{8}{3} \frac{1}{(s^2 + 4)^2} - \frac{5}{9} \frac{1}{(s^2 + 1)} \right\}$$

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2 + 2^2}$$

$$\mathcal{L}\{t \cos t\} = (-1) \frac{d}{ds} \frac{s}{s^2 + 4}$$

$$= (-1) \left[ \frac{(s^2 + 4)(1) - s \cdot (2s)}{(s^2 + 4)^2} \right]$$

$$= -1 \left[ \frac{-s^2 + 4}{(s^2 + 4)^2} \right]$$

$$= \frac{s^2 - 4}{(s^2 + 4)^2}$$



$$= \frac{5}{18} \sin 2t - \frac{2}{3} t^2 \left\{ \frac{11}{(s^2+4)^2} \right\} - \frac{5}{9} \sin 2t \quad \text{--- (A)}$$

$$s^{-1} \left\{ \frac{24}{(s^2+4)^2} \right\} \Rightarrow f(s) = \frac{2}{s^2+2} ; g(s) = \frac{2}{s^2+2}$$

$$s^{-1} \{f(s)\} = s^{-1} \left\{ \frac{2}{s^2+2} \right\} ; s^{-1} \{g(s)\} = s^{-1} \left\{ \frac{2}{s^2+2} \right\}$$

$$f(t) = \sin 2t ; g(t) = \sin 2t$$

By C.T

$$s^{-1} \{f(s) \cdot g(s)\} = \int_0^t f(u) \cdot g(t-u) du$$

$$s^{-1} \left\{ \frac{24}{(s^2+4)^2} \right\} = \int_0^t \sin 2u \sin 2(t-u) du$$

$$= \int_0^t \sin 2u \sin(2t-2u) du$$

$$= \frac{1}{2} \int_0^t 2 \sin 2u \sin(2t-2u) du$$

$$= \frac{1}{2} \int_0^t [\cos(2u-2t+2u) - \cos(2u+2t-2u)] du$$

$$= \frac{1}{2} \int_0^t [\cos(4u-2t) - \cos 2t] du$$

$$= \frac{1}{2} \left[ \frac{\sin(4u-2t)}{4} - \cos 2t \cdot u \right]_0^t$$

$$= \frac{1}{2} \left[ \frac{\sin 2t}{4} - \cos 2t \cdot t - \frac{\sin(4 \times 0 - 2t)}{4} - 0 \right]$$

$$= \frac{1}{2} \left[ \frac{\sin 2t}{4} - \cos 2t \cdot t + \frac{\sin 2t}{4} \right]$$

$$= \frac{1}{2} \left[ \frac{\sin 2t}{2} - t \cos 2t \right]$$

$$= \frac{\sin 2t}{4} - \frac{t \cos 2t}{2} \quad \text{--- Sub in (A)}$$



$$y = \frac{1}{18} \sin 2t = \frac{2}{3} \left[ \frac{\sin 2t}{4} - \frac{t \cos 2t}{4} \right] = \frac{1}{9} \sin 2t$$

49 Solve  $y'' - 2y' - 2y = 4t + e^{2t}$ ,  $y(0) = 1$ ,  $y'(0) = 1$

Given  $y'' - 2y' - 2y = 4t + e^{2t}$

Take L.T on B.S

$$L\{y''\} - 2L\{y'\} - 2L\{y\} = L\{4t + e^{2t}\}$$

$$s^2 L\{y\} - s y(0) - y'(0) - 2[s L\{y\} - y(0)] - 2L\{y\} = \frac{4}{s^2} + \frac{1}{s-2}$$

$$s^2 L\{y\} - s + 1 - 2s L\{y\} + 2 + 2L\{y\} = \frac{4}{s^2} + \frac{1}{s-2}$$

$$L\{y\} [s^2 - 2s + 2] = s + 4 = \frac{4}{s^2} + \frac{1}{s-2}$$

$$L\{y\} (s^2 - 2s + 2) = \frac{4}{s^2} + \frac{1}{s-2} + \frac{(s-4)(s^2 - 2s + 2)}{(s-2)(s-1)}$$

$$(s-2)(s-1) = \frac{4(s-3) + s^2 + (s-4)(s^2 - 2s + 2)}{s^2(s-3)}$$

$$L\{y\} \frac{(s-2)(s-1)}{(s-2)(s-1)} = \frac{4s - 12 + s^2 + s^4 - 4s^3 + 12s^2}{s^2(s-3)}$$

$$L\{y\} = \frac{s^4 - 4s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-2)(s-1)}$$

$$y = L^{-1} \left\{ \frac{s^4 - 4s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-2)(s-1)} \right\}$$

$$\frac{s^4 - 4s^3 + 13s^2 + 4s - 12}{s^2(s-3)(s-2)(s-1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1} + \frac{E}{s-2}$$



$$\begin{aligned}
 S^4 - 4S^3 + 12S^2 + 11S + 12 &= A(S-3)(S-1)(S-2) + B(S-3)(S-1)(S+2) \\
 &\quad + C(S^2)(S-1)(S-2) + D(S^2)(S-3)(S-2) \\
 &\quad + E(S^2)(S-3)(S-1)
 \end{aligned}$$

$$S=0 \Rightarrow$$



Note  $\mathcal{L}^{-1}\{f(s)\} = f(t)$

W.K.T  $\mathcal{L}\{t f(t)\} = (-1) \frac{d}{ds} f(s)$

$$t f(t) = \mathcal{L}^{-1}\left\{(-1) \frac{d}{ds} f(s)\right\}$$

$$f(t) = -\frac{1}{t} \mathcal{L}^{-1}\left\{\frac{d}{ds} f(s)\right\}$$

$$\boxed{\therefore \mathcal{L}^{-1}\{f(s)\} = -\frac{1}{t} \mathcal{L}^{-1}\{f'(s)\}}$$

$$\therefore \mathcal{L}^{-1}\{f(s)\} = -\frac{1}{t} \mathcal{L}^{-1}\{f'(s)\}$$

Find  $f(t)$

$$\mathcal{L}^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\}$$

$$f(s) = \log\left(1+\frac{1}{s^2}\right) = \log\left(\frac{s^2+1}{s^2}\right) = \log(s^2+1) - \log s^2$$

$$f(s) = \log(s^2+1) - 2\log s$$

$$f'(s) = \frac{1}{(s^2+1)} \cdot 2s - \frac{2}{s}$$

$$\mathcal{L}^{-1}\left\{\log\left(\frac{s^2+1}{s^2}\right)\right\} = -\frac{1}{t} \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s}\right\}$$

$$= -\frac{1}{t} \{2\cos t - 2\}$$

$$\text{Find } \mathcal{L}^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}$$

$$f(s) = \log\left(\frac{s+1}{s-1}\right) = \log(s+1) - \log(s-1)$$

$$f'(s) = \frac{1}{(s+1)} - \frac{1}{(s-1)}$$

$$\boxed{\mathcal{L}^{-1}\{f(s)\} = -\frac{1}{t} \mathcal{L}^{-1}\{f'(s)\}}$$

$$\mathcal{L}^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\} = -\frac{1}{t} \mathcal{L}^{-1}\left\{\frac{1}{s+1} - \frac{1}{s-1}\right\}$$

$$= -\frac{1}{t} \{e^{-t} - e^t\}$$



$$f(s) = \log \frac{(s^2+1)}{(s-1)^2} = \log(s^2+1) - \log(s-1)^2 = \log(s^2+1) - 2\log(s-1)$$

$$f'(s) = \frac{2s}{s^2+1} - 2 \cdot \frac{1}{s-1}$$

$$\mathcal{L}^{-1}\{f(s)\} = -\frac{1}{t} \mathcal{L}^{-1}\{f'(s)\}$$

$$= -\frac{1}{t} \mathcal{L}^{-1}\left\{\frac{2s}{s^2+1} - \frac{2}{s-1}\right\}$$

$$= -\frac{1}{t} [2\cos t - 2e^t]$$

Q1

Find  $\mathcal{L}^{-1}\left\{\tan^{-1}\frac{a}{s} + \cot^{-1}\frac{s}{b}\right\}$

$$f(s) = \tan^{-1}\frac{a}{s} + \cot^{-1}\frac{s}{b}$$

$$= \frac{1}{1+\frac{a^2}{s^2}} \left(-\frac{a}{s^2}\right) - \frac{1}{1+\frac{s^2}{b^2}} \cdot \frac{1}{b}$$

$$= \frac{\frac{1}{s^2+a^2} \left[-\frac{a}{s^2}\right] - \frac{1}{\frac{b^2+s^2}{b^2}} \cdot \frac{1}{b}}{s^2+a^2}$$

$$= \frac{\frac{s^2}{s^2+a^2} \left(-\frac{a}{s^2}\right) - \frac{b^2}{b^2+s^2} \cdot \frac{1}{b}}{s^2+a^2}$$

$$= \frac{-\frac{a}{s^2+a^2} - \frac{b}{s^2+b^2}}{s^2+a^2}$$

$$\frac{d}{dx} \cot^{-1} x = \frac{-1}{1+x^2}$$

$$\mathcal{L}^{-1}\{f(s)\} = -\frac{1}{t} \mathcal{L}^{-1}\{f'(s)\}$$

$$\mathcal{L}^{-1}\left\{\tan^{-1}\frac{a}{s} + \cot^{-1}\frac{s}{b}\right\}$$

$$= -\frac{1}{t} \mathcal{L}^{-1}\left\{\frac{-a}{s^2+a^2} - \frac{b}{s^2+b^2}\right\}$$

$$= -\frac{1}{t} [\sin at + \sin bt]$$

Q2

$\mathcal{L}^{-1}\{\cot^{-1} s\}$

$$f(s) = \cot^{-1} s$$

$$f'(s) = \frac{-1}{1+s^2}$$

$$\boxed{\mathcal{L}^{-1}\{f(s)\} = -\frac{1}{t} \mathcal{L}^{-1}\{f'(s)\}}$$

$$\mathcal{L}^{-1}\left\{\cot^{-1} s\right\} = -\frac{1}{t} \mathcal{L}^{-1}\left\{\frac{-1}{s^2+1}\right\}$$

$$= \frac{1}{t} \sin t = \frac{\sin t}{t}$$



Q. Find the Inverse Laplace Transform of  $\tan^{-1}(s)$

soln  $f(s) = \tan^{-1}(s)$

$$f'(s) = \frac{1}{1+(s)^2} \left[ \frac{d}{ds} \tan^{-1}(s) \right] = \frac{1}{1+s^2} \cdot \frac{1}{1+s^2} = \frac{1}{(1+s^2)^2}$$

$$\boxed{\mathcal{L}\{f'(s)\} = -\frac{1}{t} \mathcal{L}\{f(s)\}}$$

$$\mathcal{L}\{\tan^{-1}(s)\} = -\frac{1}{t} \mathcal{L}\left\{\frac{1}{(1+s^2)^2}\right\} = -\frac{1}{t} \sin 2t$$

Periodic Function: Let  $f(t)$  be the periodic function 'T' then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$$

Q. Find Laplace Transform of wave form  $f(t) = \frac{2t}{3}$  ( $0 \leq t \leq 3$ )

soln Let  $\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} \frac{2t}{3} dt$

$$= \frac{2}{3(1-e^{-3s})} \left[ \int_0^3 e^{-st} t dt \right]$$

$$\boxed{\int u dv = uv - \int v du}$$

$$u = t$$

$$= \frac{2}{3(1-e^{-3s})} \left[ t \frac{e^{-st}}{-s} - (1) \frac{e^{-st}}{(-s)^2} \right]_0^3$$

$$= \frac{2}{3(1-e^{-3s})} \left[ \frac{3e^{-3s}}{-s} - \frac{e^{-3s}}{s^2} - 0 + \frac{1}{s^2} \right]$$



Find Laplace Transform of periodic function  
 $0 \leq t \leq 2\pi$

Let  $f(t) = e^t, 0 \leq t \leq 2\pi$

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} f(t) dt$$

$$\mathcal{L}\{e^t\} = \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} e^t dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{t(1-s)} dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[ \frac{e^{t(1-s)}}{1-s} \right]_0^{2\pi}$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[ \frac{e^{2\pi(1-s)}}{1-s} - \frac{e^{0(1-s)}}{1-s} \right]$$

$$= \frac{1}{(1-s)(1 - e^{-2\pi s})} [e^{2\pi(1-s)} - e^0]$$

2nd

II nd Shifting theorem:

If  $\mathcal{L}\{f(t)\} = f(s) \cdot g(t) = \begin{cases} f(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases}$  then

$$\Rightarrow \mathcal{L}\{g(t)\} = e^{-as} f(s)$$

$$\Rightarrow \mathcal{L}^{-1}\{e^{-as} f(s)\} = \begin{cases} f(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases}$$



Find  $\mathcal{L}^{-1} \left\{ \frac{e^{-as}}{s^2+1} + \frac{e^{-s}}{s^3} \right\}$  ✓

see  $e^{-s}$  in D.T.

Let

$$\mathcal{L}^{-1} \left\{ e^{-as} \frac{1}{s^2+1} \right\}$$

$$\text{Let } f(s) = \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1} \{ f(s) \} = \mathcal{L}^{-1} \left\{ \frac{1}{s^2+1} \right\}$$

$$f(t) = \sin t$$

W.K.T

$$\mathcal{L}^{-1} \{ e^{-as} f(s) \} = \begin{cases} f(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases}$$

$$\mathcal{L}^{-1} \left\{ e^{-as} \frac{f(s)}{s^2+1} \right\} = \begin{cases} \sin(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases}$$

$$\therefore \mathcal{L}^{-1} \left\{ \frac{e^{-as}}{s^2+1} + \frac{e^{-s}}{s^3} \right\} = \begin{cases} \sin(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases} + \begin{cases} \frac{(t-1)^2}{2} & \forall t > a \\ 0 & \forall t < a \end{cases}$$

$$\mathcal{L}^{-1} \left\{ e^{-s} \frac{1}{s^3} \right\}$$

$$\text{Let } f(s) = \frac{1}{s^3} = \frac{1}{2} \frac{2}{s^3+1}$$

$$\mathcal{L}^{-1} \{ f(s) \} = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^3+1} \right\}$$

$$f(t) = \frac{1}{2} t^2$$

W.K.T

$$\mathcal{L}^{-1} \{ e^{-as} f(s) \} = \begin{cases} f(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases}$$

$$\mathcal{L}^{-1} \left\{ e^{-s} \frac{1}{s^3} \right\} = \begin{cases} \frac{(t-1)^2}{2} & \forall t > a \\ 0 & \forall t < a \end{cases}$$

29 Find  $\mathcal{L}^{-1} \left\{ \frac{e^{-s}}{s+1} \right\}$  ✓

sd =  $f(s) = \frac{1}{s+1}, a=1$

$$\mathcal{L}^{-1} \{ f(s) \} = \mathcal{L}^{-1} \left\{ \frac{1}{s+1} \right\}$$

$$f(t) = e^{-t}$$

W.K.T

$$\mathcal{L}^{-1} \{ e^{-as} f(s) \} = \begin{cases} f(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases}$$

$$\mathcal{L}^{-1} \left\{ e^{-s} \frac{1}{s+1} \right\} = \begin{cases} e^{-(t-1)} & \forall t > a \\ 0 & \forall t < a \end{cases}$$



$$\mathcal{L}^{-1} \left\{ \frac{e^{-2s}}{s^2-4} \right\}$$

$$\sin f(s) = \frac{1}{s^2-4} = \frac{1}{2} \frac{2}{s^2-2^2}$$

$$\mathcal{L}^{-1} \left\{ \frac{1}{s^2-4} \right\} = \frac{1}{2} \mathcal{L}^{-1} \left\{ \frac{2}{s^2-2^2} \right\}$$

$$f(t) = \frac{1}{2} \sinh 2t$$

W.K.T

$$\mathcal{L}^{-1} \{ e^{-as} f(s) \} = \begin{cases} f(t-a) & \forall t > a \\ 0 & \forall t < a \end{cases}$$

$$\mathcal{L}^{-1} \left\{ e^{-2s} \frac{1}{s^2-4} \right\} = \begin{cases} \frac{1}{2} \sinh 2(t-2) & \forall t > 2 \\ 0 & \forall t < 2 \end{cases}$$

P.S

$$\downarrow$$

$$\textcircled{10}$$

$$\textcircled{5} + \textcircled{5}^E$$

Laplace