28/06/2024. # Unit-II. Differential equations of Higher Order

→ An equation of the form $\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + k_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots k_n \frac{dy}{dx} + k_n y = f(x)$ is called linear differential equation of order n. —①

eqⁿ ① can be written as $[D^n + k_1 D^{n-1} + k_2 D^{n-2} + \dots + k_{n-1} D + k_n] y = f(x)$.

where $D^n = \frac{d^n}{dx^n}$, $D^{n-1} = \frac{d^{n-1}}{dx^{n-1}}$ --- $D = \frac{d}{dx}$.

General Solution;

y = complementary function + Particular integral.

$y = C.F + P.I$.

$y = y_c + y_p$.

→ Auxillary equation:

$m^n + k_1 m^{n-1} + k_2 m^{n-2} + k_3 m^{n-3} + \dots k_{n-1} m + k_n = 0$ is called an auxillary equation.

→ Working Rule for complementary function:

Solve Auxillary equation to get roots as $m_1, m_2, \dots m_n$.

Case ①: If roots are real and distinct then

$$C.F = y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \dots c_n e^{m_n x}$$

Case ②: If roots are real/equal then

m_1, m_1, m_2, m_2, m_2

then $C.F = y_c = [c_1 + x c_2] e^{m_1 x} + [c_3 + c_4 x + c_5 x^2] e^{m_3 x}$

Case ③: If roots are pair conjugate and complex i.e $\alpha \pm i\beta$ then $C.F = y_c = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

Case ④: If roots are repeated complex then i.e $\alpha \pm i\beta$, $\alpha \pm i\beta$ then complementary function

$$y_c = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x].$$

Case ⑤: If roots are pair irrational number i.e $\alpha \pm \sqrt{\beta}$. then complementary function $\Rightarrow y_c = e^{\alpha x}[c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x]$.

→ Solution of linear Homogeneous differential equations with constant coefficients:

1Q. Solve $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.

Sol: Given $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.

$\Rightarrow [D^3 - 7D - 6]y = 0$.

Auxillary equation $m^3 - 7m - 6 = 0$.

$m = -1, -2, 3$.

Complementary function $\Rightarrow y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$

$= c_1 e^{-1x} + c_2 e^{-2x} + c_3 e^{3x}$.

General Solution $y = y_c + y_p$.

$y = c_1 e^{-1x} + c_2 e^{-2x} + c_3 e^{3x}$.

2Q. Solve $4y''' + 4y'' + y' = 0$.

Sol: Given $4y''' + 4y'' + y' = 0$

$\Rightarrow [4D^3 + 4D^2 + D]\, y = 0$

Auxillary eqⁿ $4m^3 + 4m^2 + m = 0$

$m = 0, -\frac{1}{2}, -\frac{1}{2}$.

C.F $\Rightarrow$ $y_c = c_1 e^{m_1 x} + (c_2 + c_3 x) e^{m_2 x}$.

$= c_1 e^{0x} + (c_2 + c_3 x) e^{-\frac{1}{2}x}$.

General Solution: $y = y_c + y_p$

$y = c_1 + (c_2 + c_3 x) e^{-\frac{1}{2}x}$.

V.imp. 16/2 Marks

Working Rule to find Particular Integral.

Type ①: If RHS $= e^{ax}$ i.e $f(D)y = e^{ax}$ then

$y_p = \frac{1}{f(D)} e^{ax}$ Replace D by a.

If $f(a) = 0 \Rightarrow y_p = \frac{x}{1!} \frac{1}{f'(D)} e^{ax}$ $(D = a)$

If $f'(a) = 0 \Rightarrow y_p = \frac{x^2}{2!} \frac{1}{f''(D)} e^{ax}$

If $f''(a) = 0 \Rightarrow y_p = \frac{x^3}{3!} \frac{1}{f'''(D)} e^{ax}$

A.eqⁿ → C.F → P.I → General Solution

1Q. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = e^{3x}$

Sol: Given $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = e^{3x}$.

$\Rightarrow (D^3 - 3D^2 + 4)y = e^{3x}$.

Auxillary eq^n $= m^3 - 3m^2 + 4$

$m = -1, 2, 2$

C.F $\Rightarrow y_c = c_1 e^{-1x} + [c_2 + c_3 x] e^{2x}$

P.I $\Rightarrow y_p = \frac{1}{f(D)} e^{ax} \rightarrow e^{3x}$.

$= \frac{1}{D^3 - 3D^2 + 4} e^{3x}$

Replace $D = a = 3$.

$= \frac{1}{3^3 - 3(3)^2 + 4}$

$= \frac{1}{4} e^{3x}$

General Solution $y = y_c + y_p$.

$y = c_1 e^{-1x} + [c_2 + c_3 x] e^{2x} + \frac{1}{4} e^{3x}$.

2Q. Solve $y'' - y' - 2y = 3e^{2x}$

Sol: Given $y'' - y' - 2y = 3e^{2x}$

$\Rightarrow [D^2 - D - 2]y = 3e^{2x}$

Auxillary equation $m^2 - m - 2 = 0$

$m = 2, -1$

C.F $\Rightarrow y_c = c_1 e^{-1x} + c_2 e^{2x}$

P.I $\Rightarrow y_p = \frac{1}{D^2-D-2} 3e^{2x}$

$= \frac{1}{(2)^2-2-2} 3e^{2x}$ Replace $D=a=2$.

$= \frac{1}{0} 3e^{2x}$

$\boxed{\Rightarrow \frac{x}{1!} \frac{1}{f'(D)} e^{ax}}$

$\Rightarrow \frac{x}{1!} \frac{1}{2D-1} 3e^{2x}$

Now replace $D=a=2$.

$\Rightarrow \frac{x}{1} \times \frac{1}{2(2)-1} 3e^{2x}$

$\Rightarrow x \times \frac{1}{\cancel{3}} \times \cancel{3}e^{2x}$

$\Rightarrow xe^{2x}$.

General Solution $y = y_c + y_p$

$y = c_1e^{-x} + c_2e^{2x} + xe^{2x}$.

3Q. Solve $\frac{d^3y}{dx^3} - y = [e^x + e^{-x}]^2$.

Sol. Given $\frac{d^3y}{dx^3} - y = [e^x + e^{-x}]^2$

$(D^3-1)y = [e^x + e^{-x}]^2$

Auxillary eq^n $m^3-1=0$

$m = 1, \frac{-1}{2} + \frac{\sqrt{3}}{2} i,$

Complementary function $\Rightarrow y_c = c_1e^{1x} + e^{-1/2x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$

P.I $\Rightarrow y_p = \frac{1}{D^3-1}[e^x+e^{-x}]^2$

$\because [e^x+e^{-x}]^2$

$= e^{2x}+2e^x \cdot e^{-x}+e^{-2x}$

$= e^{2x}+2+e^{-2x}$

$= \frac{1}{D^3-1}[e^{2x}+e^{-2x}+2]$

$e^{0x}=1$

$2(1)=2$

$= \frac{1}{D^3-1}e^{2x}+\frac{1}{D^3-1}e^{-2x}+\frac{1}{D^3-1}2e^{0x}$

Replace $D = a = 2$

$= \frac{1}{8-1}e^{2x}+\frac{1}{-8-1}e^{-2x}+\frac{1}{0-1}2$

$= \frac{e^{2x}}{7}-\frac{e^{-2x}}{9}-2$

General Solution $y = y_c + y_p$

$y = c_1e^{1x}+e^{-1/2x}\left[c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}\right]+$

$\frac{e^{2x}}{7}-\frac{e^{-2x}}{9}-2$

H/w:

Q4. Solve $(D^2+6D+9)y = e^{-3x}$

Q5. Solve $(D^3+3D^2+D)y = e^{2x}$

Q6. Solve $(D^3-1)y = (e^x+1)^2$

→ Type ②: If $f(D)y = \sin ax / \cos ax$ then

$P.I \Rightarrow y_p = \frac{1}{f(D)} \sin ax / \cos ax$.

Replace $D^2 = -a^2$ $\boxed{\text{not } (-a)^2}$

If $f(-a^2) = 0$ then $y_p = \frac{x}{1!} \frac{1}{f'(D)} \sin ax / \cos ax$.

Q1. Solve $[D^2+9]y = \cos 3x + \sin 3x + \sin 2x$.

Sol. Given $[D^2+9]y = \cos 3x + \sin 3x + \sin 2x$

$\sin ax$
↓
$\sin 3x$
↓
$a = 3$

Auxillary eqⁿ $m^2+9=0$

$m = 0 \pm 3i$.

$C.F \Rightarrow y_c = e^{0x}[c_1\cos 3x + c_2 \sin 3x]$

$= c_1 \cos 3x + c_2 \sin 3x$

$P.I \Rightarrow y_p = \frac{1}{D^2+9}[\cos 3x + \sin 3x + \sin 2x]$

Replace D^2 by $-a^2$

$y_p = \frac{1}{D^2+9}\cos 3x + \frac{1}{D^2+9}\sin 3x + \frac{1}{D^2+9}\sin 2x$

$a = 3, a^2 = 9$

$= \frac{1}{-9+9}\cos 3x + \frac{1}{-9+9}\sin 3x + \frac{1}{-4+9}\sin 2x$

$= \frac{1}{0}\cos 3x + \frac{1}{0}\sin 3x + \frac{1}{5}\sin 2x$

↓

$= \frac{x}{1}\frac{1}{2D}\cos 3x + \frac{x}{1}\frac{1}{2D}\sin 3x + \frac{\sin 2x}{5}$

Differential D integrate dx at we get 1.

↓

$= \frac{x}{2}\int \cos 3x\,dx + \frac{x}{2}\int \sin 3x\,dx + \frac{\sin 2x}{5}$

$= \frac{x}{2}\frac{\sin 3x}{3} + \frac{x}{2}\frac{(-\cos 3x)}{3} + \frac{\sin 2x}{5}$.

$\therefore$ General Solution $y = y_c + y_p$.

$$y = c_1\cos 3x + c_2\sin 3x + \frac{x}{6}\sin 3x - \frac{x}{6}\cos 3x + \frac{\sin 2x}{5}.$$

Q2. Solve $[D^3-1]y = e^x + \sin 3x + 2$.

Sol: Given $[D^3-1]y = e^x + \sin 3x + 2$.

Auxillary eq^n $m^3-1=0$

$$m = 1, -\frac{1}{2} \pm \frac{\sqrt{3}i}{2}.$$

$$C.F \Rightarrow y_c = c_1e^{1x} + e^{-\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right]$$

$$P.I \Rightarrow y_p = \frac{1}{(D^3-1)}[e^x + \sin 3x + 2].$$

$\because e^{0x} = 1$.

$2e^{0x} = 2(1) = 2$.

$$y_p = \frac{e^x}{(D^3-1)} + \frac{\sin 3x}{(D^3-1)} + \frac{2e^{0x}}{(D^3-1)}$$

Replace D^2 by $-a^2$; D by a, ; D by a

$a=3,\ a^2=9$; $D=a=1$; $D=a=0$

$$= \frac{1}{1-1}e^x + \frac{1}{D\cdot D^2-1}\sin 3x + \frac{1}{0-1}2e^{0x}$$

$$= \frac{x}{1!}\frac{1}{3D^2}e^x + \frac{1}{-9D-1}\sin 3x - 2$$

$$= \frac{x}{1} \times \frac{1}{3(1)^2}e^x + \frac{1}{-9D-1}\sin 3x - 2$$

$$= \frac{xe^x}{3} - \frac{1}{9D+1}\sin 3x - 2$$

$$= \frac{xe^x}{3} - \frac{1}{9D+1}\frac{(9D-1)}{(9D-1)}\sin 3x - 2. \quad (\because \text{Rationalise})$$

$$= \frac{xe^x}{3} - \frac{(9D-1)\sin 3x}{81D^2-1} - 2$$

$$\Rightarrow \frac{xe^x}{3} - \frac{9D\sin 3x - \sin 3x}{81(-9)-1} - 2 \qquad (D^2 = -a^2 = -9).$$

$$(\because \text{differentiated})$$

$$\Rightarrow \frac{xe^x}{3} - \frac{9\cos 3x \cdot 3 - \sin 3x}{-729-1} - 2$$

$$\Rightarrow \frac{xe^x}{3} + \frac{27\cos 3x - \sin 3x}{+730} - 2$$

General Solution : $y = y_c + y_p$

$$y = c_1 e^x + e^{-1/2 x}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2} x\right] +$$

$$\frac{xe^x}{3} + \frac{27\cos 3x - \sin 3x}{730} - 2.$$

01/07/2024.

Q. Solve $y''+4y'+4y=4\cos x+3\sin x$.

Sol: Given $[D^2+4D+4]y = 4\cos x+3\sin x$.

Auxillary equation $m^2+4m+4=0$

$m=-2,-2$.

Complementary function $y_c=[c_1+xc_2]e^{-2x}$

Particular Integral : $y_P = \frac{1}{[D^2+4D+1]}[4\cos x+3\sin x]$

$PI_1 = \frac{1}{[D^2+4D+4]} \times 4\cos x$.

Replace $D^2=a^2=-1$.

$= \frac{1}{-1+4D+4} \times 4\cos x$.

$= \frac{1}{4D+3} 4\cos x$.

Rationalise

$= \frac{(4D-3)}{(4D+3)(4D-3)} \times 4\cos x$

$= \frac{(4D-3)(4\cos x)}{16D^2-9}$

$= \frac{16D\cos x-12\cos x}{16(-1)-9}$

$= \frac{16(-\sin x)-12\cos x}{-25}$

$PI_1 = \frac{-16\sin x-12\cos x}{-25}$.

$P.I_2 = \frac{1}{[D^2+4D+4]} \times 3\sin x$

Replace $D^2=a^2=-1$

$= \frac{1}{-1+4D+4} \times 3\sin x$

$= \frac{1}{4D+3} \times 3\sin x$

Rationalise.

$= \frac{(4D-3)}{(4D+3)(4D-3)} \times 3\sin x$

$= \frac{(4D-3)\times 3\sin x}{16D^2-9}$

$= \frac{12D\sin x-9\sin x}{16D^2-9}$

$= \frac{12(\cos x)-9\sin x}{-25}$

$PI_2 = \frac{12\cos x-9\sin x}{-25}$.

Particular Integral = $P.I_1 + P.I_2$.

$$= \frac{16\sin x + 12\cos x}{25} + \frac{[12\cos x - 9\sin x]}{-25}$$

$$= \frac{16\sin x + 12\cos x}{25} - \left(\frac{12\cos x - 9\sin x}{25}\right)$$

$$= \frac{16\sin x + 9\sin x + 12\cos x - 12\cos x}{25}$$

$$= \frac{\cancel{25}\sin x}{\cancel{25}}$$

$= \sin x$

General solution $y = y_c + y_p$.

$$= (c_1 + c_2 x) e^{-2x} + \sin x.$$

★ **Note:**

① $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

② $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

③ $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

④ $2\sin A\sin B = \underset{\cos}{\cancel{\sin}}(A-B) - \underset{\cos}{\cancel{\sin}}(A+B)$

If multiple functions are there then use these formulae.

Q4. Solve $[D^2 - 4D + 3]y = \sin 3x \cdot \cos 2x$.

Sol. Given $[D^2 - 4D + 3]y = \sin 3x \cos 2x$

Auxillary function: $m^2 - 4m + 3 = 0$

$m = 1, 3$.

Complementary function: $y_c = c_1 e^{1x} + c_2 e^{3x}$.

$P.I = y_P = \frac{1}{D^2-4D+3} \sin 3x \cos 2x$

[Multiply & divide by 2]

$y_P = \frac{1}{2(D^2-4D+3)} 2\sin 3x \cos 2x$

$y_P = \frac{1}{2[D^2-4D+3]} [\sin(3x+2x) + \sin(3x-2x)]$

$= \frac{1}{2[D^2-4D+3]} [\sin 5x + \sin x]$

$PI_1 = \frac{1}{[D^2-4D+3]} \sin 5x$

$\Rightarrow D^2 = -a^2 = -5^2 = -25.$

$= \cancel{\frac{1}{-2D}} \frac{1}{-25-4D+3} \sin 5x$

$= \frac{1}{-4D-22} \sin 5x$

$= \frac{\pm 1}{-(4D+22)} \sin 5x$

$= \frac{-(4D-22)}{(4D+22)(4D-22)} \sin 5x$

$= \frac{-4D\sin 5x - 22\sin 5x}{16D^2-484}$

$= \frac{-4(\cos 5x \cdot 5) - 22\sin 5x}{16(-25)-484}$

$= \frac{-20\cos 5x - 22\sin 5x}{-884}$

$PI_1 = \frac{\cancel{-}(20\cos 5x + 22\sin 5x)}{\cancel{-}884}$

$P.I_2 = \frac{1}{[D^2-4D+3]} \sin x$

$\Rightarrow D^2 = -a^2 = -5^2 = -\cancel{25} -1$

$\Rightarrow \frac{1}{-1-\cancel{25}-4D+3} \sin x$

$\Rightarrow \frac{1}{-4D-\cancel{22}\,2} \sin x$

$= \frac{1}{-(4D+2\cancel{2})} \sin x$

$= \frac{-(4D-2\cancel{2})}{(4D+2\cancel{2})(4D-2\cancel{2})} \sin x$

$= \frac{(2+4D)\sin x}{(2-4D)(2+4D)}$

$= \frac{2\sin x - 4\cos x}{4-16D^2}$

$= \frac{2\sin x - 4\cos x}{4-16(-1)}$

$PI_2 = \frac{2\sin x - 4\cos x}{20}$

$P.I = \frac{1}{2}[PI_1 + PI_2]$.

$$y_P = \frac{1}{2}\left[\frac{20\cos 5x + 22\sin 5x}{884} + \frac{2\sin x - 4\cos x}{20}\right]$$

$$y_P = \frac{\cos 5x + 11\sin 5x}{884} + \frac{\sin x - 2\cos x}{20}$$

General solution $y = y_c + y_P$.

→ Type③: If $f(D)y = x^n$

$$y_P = \frac{1}{f(D)} x^n.$$

$$= \frac{1}{[1 \pm \phi(D)]} x^n$$

$$= [1 \pm \phi(D)]^{-1} x^n$$

* $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$
* $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \cdots$

Q1. Solve $[D^2+D+1]y = x^3$.

Sol. Given $[D^2+D+1]y = x^3$.

Auxillary eq^n $m^2+m+1=0$

$m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

Complementary function $\Rightarrow y_c = e^{-1/2x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$

P.I $\Rightarrow y_p = \frac{1}{[D^2+D+1]}x^3$

$\Rightarrow y_p = \frac{1}{[1+(D^2+D)]}x^3$

$= [1+(D^2+D)]^{-1}x^3$

$= [1-(D^2+D)+(D^2+D)^2-(D^2+D)^3 +]x^3$

$= 1-\cancel{D^2}-D+\cancel{D^4}+2D^3+\cancel{D^2}-\cancel{D^6}-3\cancel{D^5}-3\cancel{D^4}-D^3]x^3$

$= [1-D+D^3]x^3 = [x^3 - x^3D + x^3D^3]$

$= x^3-3x^2+6$.

To We cancel terms based on pave. If x^3 then cancel all terms greater than 3. If x^2 then cancel all terms greater than 2 and soon.

$Dx^3 = 3x^2$

$D(3x^2) = 6x$

$D(6x) = 6$

$\therefore$ General Solution : $y_c + y_p = y$

$\therefore y = \underbrace{e^{-1/2x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]}_{y_c}$

$+ \underbrace{x^3 - 3x^2 + 6}_{y_p} = y$

Q2. Solve $[D^3-D^2-D+1]y=[1+x^2]$

Sol. Given $[D^3-D^2-D+1]\,y=[1+x^2]$

Auxillary function : $m^3-m^2-m+1=0$

$m=-1,1,1$

Complementary function $\Rightarrow y_c = c_1e^{-1x}+[c_2+c_3x]e^{1x}$

$$P.I \Rightarrow y_P = \frac{1}{[D^3-D^2-D+1]}[1+x^2]$$

$$= \frac{1}{[1+[D^3-D^2-D]]}[1+x^2]$$

$$= [1+[D^3-D^2-D]]^{-1}[1+x^2]$$

$$= [1-(D^3-D^2-D)+(D^3-D^2-D)^2+\cdots][1+x^2]$$

$$= [1-D^3+D^2+D+D^6+D^4+D^2-2D^5+2D^3-2D^4][1+x^2]$$

$-D^6$

$$= [1+D+2D^2][1+x^2]$$

$$= [1+x^2+2x+2(2)]$$

$$= 1+4+x^2+2x$$

$$y_P = x^2+2x+5$$

$$y = y_c + y_P$$

$$= c_1e^{-1x}+[c_2+c_3x]e^{1x}+x^2+2x+5$$

//

Very Imp.

83. $(D-2)^2 y = 8[e^{2x} + \sin 2x + x^2]$

Sol. Given $(D-2)^2 y = 8[e^{2x} + \sin 2x + x^2]$

Auxillary equation $(m-2)^2 = 0$

$(m-2)(m-2) = 0$

$m = 2, 2$.

Complementary function $\Rightarrow y_c = [c_1 + c_2 x]e^{2x}$

$P.I \Rightarrow y_p = \frac{1}{[D^2 - 4D + 4]} 8[e^{2x} + \sin 2x + x^2]$

$P.I_1 = \frac{1}{[D^2 - 4D + 4]} e^{2x}$

$\boxed{D = a = 2}$

$= \frac{1}{-4-8+4} e^{2x}$

$= \frac{1}{0} e^{2x}$

So; $\frac{x}{1} \cdot \frac{1}{2D-4} e^{2x} = \frac{1}{0}$

$= \frac{x^2}{2} \frac{1}{2} e^{2x} \Rightarrow \frac{x^2 e^{2x}}{4}$

$\therefore P.I_1 = \frac{x^2 e^{2x}}{4}$

$P.I_2 = \frac{1}{D^2 - 4D + 4} \sin 2x$

$\boxed{D^2 = -a^2 = -2^2 = -4}$

$= \frac{1}{-4 - 4D + 4} \sin 2x$

$= \frac{1}{-4D} \sin 2x$

$= -\frac{1}{4} \int \sin 2x \, dx$

$= -\frac{1}{4} \left(\frac{-\cos 2x}{2}\right)$

$= \cos 2x / 8$.

$\therefore P.I_2 = \frac{\cos 2x}{8}$

$$PI_3 = \frac{1}{[D^2-4D+4]} x^2$$

$$= \frac{1}{4+D^2-4D} x^2$$

$$= \frac{1}{4\left[1+\left(\frac{D^2-4D}{4}\right)\right]} x^2$$

$$= \frac{1}{4}\left[1+\left[\frac{D^2-4D}{4}\right]\right]^{-1} x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2-4D}{4} + \frac{[D^2-4D]^2}{16 \; 4^2}\right] x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} - \frac{4D}{4} + \frac{D^4}{16} + \frac{16D^2}{16} - \frac{2D^2(4D)}{\cancel{16}\cancel{4}\,2}\right] x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} - D + \frac{\cancel{D^4}}{\cancel{16}} + D^2 - \frac{\cancel{D^3}}{\cancel{2}}\right] x^2$$

$$= \frac{1}{4}\left[1 - \frac{D^2}{4} + D^2 - D\right] x^2$$

$$= \frac{1}{4}\left[x^2 - \frac{2}{4} + 2 - 2x\right]$$

$$\therefore P.I_3 = \frac{1}{4}\left[x^2 - 2x + 2 - \frac{2}{4}\right]$$

$$= \frac{1}{4}\left[x^2 - 2x + \frac{\cancel{6}^{\,3}}{\cancel{4}_{\,2}}\right]$$

$$= \frac{1}{4}\left[x^2 - 2x + \frac{3}{2}\right]$$

$$y_p = 8[PI_1 + PI_2 + PI_3]$$

$$= 8\left[\frac{x^2 e^{2x}}{4} + \frac{\cos 2x}{8} + \frac{1}{4}\left[x^2 - 2x + \frac{3}{2}\right]\right]$$

$= 2x^2e^{2x} + \cos 2x + 2x^2 - 4x + 3$

$\therefore$ General Solution is $2x^2e^{2x} + \cos 2x + 2x^2 - 4x + 3 = y_p +$

$$[c_1 + c_2x]e^{2x} = y \qquad (y_c + y_p = y).$$

→ Type ④: If $f(D) = y = e^{ax}\phi(x)$

$$y_p = \frac{1}{f(D)} e^{ax}\phi(x).$$

$$= e^{ax}.\frac{1}{f(D+a)}\phi(x) \qquad \boxed{D = D+a}$$

Remaining problem solve in type ② and type ③.

Q1. Solve $(D^2+2)y = e^x \sin x$

Sol. Given $(D^2+2)y = e^x \sin x$.

Auxillary equation $m^2+2=0$

$$m^2 = -2$$

$$m = 0 \pm \sqrt{2}i$$

Complementary function $\Rightarrow y_c = e^{0x}[c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x]$

P.I $\Rightarrow y_p = \frac{1}{D^2+2} e^x \sin x$

$$= e^x.\frac{1}{(D+1)^2+2} \sin x \qquad \boxed{D = D+a = D+1}$$

$$= e^x \frac{1}{D^2+2D+3} \sin x \qquad \boxed{D^2 = a^2 = -1}$$

$$= e^x \frac{1}{-1+2D+3} \sin x$$

$$e^{x} \frac{1}{-1+2D+3} \sin x$$

$$= e^{x} \frac{1}{(2D+2)} \sin x$$

$$= \frac{e^{x}}{2} \frac{1}{D+1} \sin x$$

$$= \frac{e^{x}}{2} \cdot \frac{(D-1)}{(D+1)(D-1)} \sin x$$

$$= \frac{e^{x}}{2} \frac{(D-1) \sin x}{D^{2}-1}$$

$$\boxed{D^{2}=a^{2}=-1}$$

$$= \frac{e^{x}}{2} \left[\frac{\cos x - \sin x}{-1-1} \right]$$

$$= \frac{e^{x}}{2} \left[\frac{\cos x - \sin x}{-2} \right]$$

$$= \frac{e^{x} [\cos x - \sin x]}{-4}$$

$$\therefore \text{P.I} = -\frac{e^{x} [\cos x - \sin x]}{4}$$

General solution : $y = y_c + y_p$

$$= e^{0x} [c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x] + \left(-\frac{e^{x}(\cos x - \sin x)}{4} \right)$$

$$= [c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x] - \frac{e^{x} (\cos x - \sin x)}{4}$$

06/07/00245.

Q2. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$.

Sol. Given eqn $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$

$\Rightarrow [D^2 - 6D + 13]y = 8e^{3x}\sin 2x$

Auxillary equation $m^2 - 6m + 13 = 0$.

$m = 3 \pm 2i$.

C.F $\Rightarrow y_c = e^{3x}[c_1\cos 2x + c_2\sin 2x]$

P.I $\Rightarrow y_p = \frac{1}{D^2 - 6D + 13} 8e^{3x}\sin 2x$

Replace

$\boxed{\begin{array}{l} D = D + a \\ D = D + 3 \end{array}}$

$= 8e^{3x} \cdot \frac{1}{(D+3)^2 - 6(D+3) + 13} \sin 2x$

$= 8e^{3x} \frac{1}{D^2 + \cancel{6D} + 9 - \cancel{6D} - 18 + 13} \sin 2x$

$= 8e^{3x} \frac{1}{D^2 + 4} \sin 2x$

Replace $D^2 = -a^2 = -2^2 = -4$

$= 8e^{3x} \cdot \frac{1}{-4 + 4} \sin 2x$

$= \frac{8e^{3x}\sin 2x}{0}$ $= 8e^{3x} \cdot \frac{x}{2D} \sin 2x$

$= \frac{8}{2} xe^{3x} \int \sin 2x \, dx$

$= 4xe^{3x}\left(-\frac{\cos 2x}{2}\right)$

$y_p = -2xe^{3x}\cos 2x$

$\therefore$ General solution $y = y_c + y_p \Rightarrow e^{3x}[c_1\cos 2x + c_2\sin 2x] - 2xe^{3x}\cos 2x$.

Q3. Solve $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}[1+x]$.

Sol: Given eq^n $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}[1+x]$.

$\Rightarrow [D^2 - 7D + 6]\, y = e^{2x}[1+x]$

Auxillary equation: $m^2 - 7m + 6 = 0$

$m = 1, 6$.

Complementary function $y_c = c_1 e^{1x} + c_2 e^{6x}$.

Particular Integral $y_p = \frac{1}{D^2 - 7D + 6} e^{2x}[1+x]$

$$y_p = e^{2x} \cdot \frac{1}{(D+2)^2 - 7(D+2) + 6}(1+x) \qquad \boxed{\begin{aligned} D &= D+a \\ &= D+2 \end{aligned}}$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 7D - 14 + 6}(1+x)$$

$$= e^{2x} \frac{1}{D^2 - 3D - 4}(1+x)$$

~~Replace $D^2 = -a^2 = -2^2 = -4$~~

$$= e^{2x} \frac{1}{-4\left[1 - \left(\frac{D^2 - 3D}{4}\right)\right]}(1+x)$$

$$= \frac{e^{2x}}{-4}\left[1 - \left(\frac{D^2 - 3D}{4}\right)\right]^{-1}(1+x)$$

expansion

$$= \frac{-e^{2x}}{4}\left[1 + \frac{\cancel{D^2} - 3D}{4}\right](1+x)$$

$$= \frac{-e^{2x}}{4}\left[1 - \frac{3D}{4}\right](1+x)$$

$\Rightarrow -\frac{e^{2x}}{4}\left[1+x-\frac{3}{4}(1)\right]$

$y_p = -\frac{e^{2x}}{4}\left[x+\frac{1}{4}\right]$

$\therefore$ General Solution $y = y_c + y_p = c_1e^{1x} + c_2e^{6x} - \frac{e^{2x}}{4}\left[x+\frac{1}{4}\right]$.

Q4- Solve $[D^3-3D^2+3D-1]y = x^2e^x$.

Sol. Given eq^n : $[D^3-3D^2+3D-1]y = x^2e^x$

Auxillary equation : $m^3-3m^2+3m-1=0$

$m = 1, 1, 1$

Complementary function : $y_c = [c_1+c_2x+c_3x^2]e^{1x}$

Particular Integral $y_p = \frac{1}{D^3-3D^2+3D-1} x^2e^x$

$= e^x \frac{1}{D^3-3D^2+3D-1} x^2$.

Replace $\boxed{D = D+a = D+1}$.

$= e^x \frac{1}{(D+a)^3-3(D+a)^2+3(D+a)-1} x^2$

$= e^x \frac{1}{(D+1)^3-3(D+1)^2+3(D+1)-1} x^2$

$= e^x \frac{1}{D^3+\cancel{3D^2}+\cancel{3D}+\cancel{1}-\cancel{3D^2}-\cancel{6D}-\cancel{3}+\cancel{3D}+\cancel{3}-\cancel{1}} x^2$

$= e^x \frac{1}{D^3} x^2$

$\boxed{\frac{1}{D^3} = \frac{1}{D^2} \cdot \frac{1}{D}}$

$= \frac{e^{x}}{D^2} \int x^2\,dx$

$e^x \frac{1}{D^2} \frac{x^3}{3}$

$= \frac{e^x}{3} \frac{1}{D} \int x^3 dx$

$= \frac{e^x}{3} \frac{1}{D} \frac{x^4}{4}$

$= \frac{e^x}{12} \int x^4 dx$

$= \frac{e^x}{12} \frac{x^5}{5}$

$= \frac{e^x x^5}{60}$ //

H/w: Solve: $(D^2+4D+8)y = e^{-2x}\sin 3x \sin x$.

Sol: Auxillary eq^n: $m^2+4m+8=0$ $m = -2+2i, -2-2i$

complementary funct^n $y_c = e^{-2x}[c_1\cos 2x + c_2 \sin 2x]$

P.I = $y_p = \frac{1}{D^2+4D+8} e^{-2x}\sin 3x \sin x$.

$a=2$ $(D = D+a)$

$y_p = \frac{1}{(D-2)^2+4(D-2)+8} e^{-2x} \sin 3x \sin x$

$= \frac{1}{D^2+4-\cancel{4D}+\cancel{4D}-\cancel{8}+\cancel{8}} e^{-2x}\sin 3x \sin x$

$= e^{-2x} \frac{1}{D^2+4} \sin 3x \sin x$.

$= e^{-2x} \frac{1}{D^2+4} [\frac{1}{2} 2\sin 3x \sin x]$

$= \frac{1}{2} e^{-2x} \frac{1}{D^2+4} [\sin x - \sin 4x]$

$2\sin a \sin b = \sin(a+b) - \sin(a+b)$

$\boxed{2\sin a \sin b = \cos(a-b) - \cos(a+b)}$

$\Rightarrow \frac{1}{2} e^{-2x} \left(\frac{1}{D^2+4} \sin x \right) - \left(\frac{\sin 4x}{D^2+4} \right)$

Replace D^2 by $-a^2$

$\boxed{D^2 = -a^2 = -1}$

$D^2 = -a^2 = -4^2 = -16$

$\frac{1}{2} e^{-2x} \frac{1}{3} \sin x - \frac{\sin 4x}{-12}$

$= \frac{e^{-2x}}{2} \left(\frac{\sin x}{3} + \frac{\sin 4x}{12} \right) = y_p$

$\therefore$ General solution: $y = y_c + y_p$.

$y = e^{-2x}\sin 3x \sin x + \frac{e^{-2x}}{2}\left(\frac{\sin 3x}{3} + \frac{\sin 4x}{12} \right)$

→ Type (5) : If $f(D)y = x\phi(x)$.

→ Particular Integral ⇒ $y_p = \frac{1}{f(D)} x\phi(x)$.

$$= \left[\frac{x}{f(D)} - \frac{f'(D)}{[f(D)]^2}\right]\phi(x)$$

Remaining problems solve in type (2).

Q1. Solve $[D^2-1]y = x\sin 3x + \cos x$

Sol: Given equation : $[D^2-1]y = x\sin 3x + \cos x$.

Auxillary eq^n : $m^2-1=0$

$m = \pm 1$.

C.F ⇒ $y_c = c_1e^{1x} + c_2e^{-1x}$.

P.I ⇒ $y_p = \frac{1}{D^2-1}[x\sin 3x + \cos x]$

$y_p = \frac{1}{D^2-1}x\sin 3x + \frac{1}{D^2-1}\cos x$

↓ type 5 → type 2. $D^2 = a^2 = -1^2$.

$$= \left[\frac{x}{(D^2-1)} - \frac{2D}{(D^2-1)^2}\right]\sin 3x + \frac{1}{(-1-1)}\cos x$$

$$= \frac{x}{D^2-1}\sin 3x - \frac{2D}{(D^2-1)^2}\sin 3x - \frac{1}{2}\cos x$$

↳ differentiate $\sin 3x = \cos 3x(3) = 3\cos 3x$.

$$= \frac{x}{-9-1}\sin 3x - \frac{2\cos 3x(3)}{(-9-1)^2} - \frac{1}{2}\cos x$$

$$y_p = \frac{x}{-10}\sin 3x - \frac{6\cos 3x}{100} - \frac{1}{2}\cos x.$$

∴ General Solution : $y = y_c + y_p$

$$c_1e^{x} + c_2e^{-x} - \frac{x}{10}\sin 3x - \frac{6\cos 3x}{100} - \frac{1}{2}\cos x.$$

Q2. Solve $[D^2-2D+1]y = xe^x\sin x$.

Sol: Given $[D^2-2D+1]y = xe^x\sin x$ 	Vimp C.F

Auxillary eq^n : $m^2-2m+1=0$

$m = 1, 1$. 	(degree = constants).

$\Downarrow$

$2 = c_1, c_2$

$3 = c_1, c_2, c_3$.

C.F $\Rightarrow$ $y_c = [c_1 + c_2 x]e^{1x}$.

$\Rightarrow$ P.I $\Rightarrow$ $y_p = \dfrac{1}{D^2-2D+1} e^x[x\sin x]$

$= e^x \dfrac{1}{D^2-2D+1} x\sin x \rightarrow$ Type (4)

Replace $\boxed{D = D+a = D+1}$

$= e^x \dfrac{1}{(D+1)^2-2(D+1)+1} x\sin x$

$= e^x \dfrac{1}{D^2+2D+1-2D-2+1} x\sin x$

$= e^x \dfrac{1}{D^2}[x\sin x]$ (go for type 5).

$= e^x \left[\dfrac{x}{D^2} - \dfrac{2D}{(D^2)^2} \right] \sin x$

$= e^x \left[\dfrac{x}{D^2}\sin x - \dfrac{2D}{(D^2)^2}\sin x \right]$

$= e^x \left[\dfrac{x}{(-1)}\sin x - \dfrac{2\cos x}{(-1)^2} \sin x \right]$

$= e^x[-x\sin x - 2\cos x]$

$\therefore$ General solution is $y = [c_1 + c_2 x]e^x + e^x[-x\sin x - 2\cos x]$ //

Q3- Solve $[D^2-1]y = x\sin x + x^2 e^x$.

Sol- Given $[D^2-1]y = x\sin x + x^2e^x$.

Auxillary eqⁿ: $m^2-1=0$

$m = +1, -1$.

C.F $\Rightarrow y_c = [c_1e^{1x} + c_2e^{-1x}]$.

$\Rightarrow$ P.I $\Rightarrow y_p = \frac{1}{D^2-1} x\sin x + x^2e^x$.

$= \frac{x\sin x}{D^2-1} + \frac{x^2e^x}{D^2-1}$

$P.I_1 = \frac{1}{D^2-1} x\sin x$.

(type 5)

$= \left| \frac{x}{D^2-1} - \frac{2D}{(D^2-1)^2} \right| \sin x$

$= \frac{x}{D^2-1}\sin x - \frac{2D}{(D^2-1)^2}\sin x$.

$= \frac{x}{(-1)-1}\sin x - \frac{2D}{(-1-1)^2}\sin x$

$= -\frac{x}{2}\sin x - \frac{2D}{4_2}\sin x$.

$= -\frac{x}{2}\sin x - \frac{D}{2}\sin x$ diff sin x = cos x

$= -\frac{x\sin x}{2} - \frac{\cos x}{2}$

$= -\frac{x}{2}\sin x - \frac{\cos x}{2}$

$P.I_2 = \frac{1}{(D^2-1)} e^x x^2$

$= \frac{e^x}{[(D+1)^2-1]} x^2$

$D = D+a$
$= D+1$

$= e^x \frac{1}{D^2+2D+1-1} x^2$

$= e^x \frac{1}{2D(1+\frac{D}{2})} x^2$

$= \frac{e^x}{2D}\left[1+\frac{D}{2}\right]^{-1} x^2$

expansion.

$= \frac{e^x}{2D}\left[1 - \frac{D}{2} + \frac{D^2}{4} + \cdots\right]x^2$

$= \frac{e^x}{2D}\left[x^2 - \frac{2x}{2} + \frac{2}{4} + \cdots\right]$

$D x^2 = 2x$
$D(2x) = 2$

$= \frac{e^x}{2}\int \left(x^2 - x + \frac{1}{2}\right)dx$

$= \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right]$

$y_p = P.I_1 + P.I_2$

$= -\frac{x}{2}\sin x - \frac{\cos x}{2} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{1}{2}x\right]$

$\therefore$ General Solution $y = y_c + y_p$.

$$y = [c_1 e^{1x} + c_2 e^{-1x}] - \frac{x}{2}\sin x - \frac{\cos x}{2} + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}\right].$$

H/w.

Q. Solve $y'' + 4y = x\cos x$ (type 1, 5)

Q. Solve $(D^2+9)y = x\sin 3x + e^{2x}$

v.imp

(2) Cauchy's Euler Differential Equation.

An equation of the type $[x^n D^n + k_1 x^{n-1} D^{n-1} + k_2 x^{n-2} D^{n-2} + \cdots k_{n-1} x D] y = f(x)$ — (1)

is called Cauchy's Euler Differential equation.

Working Rule:

(1) $\log_e x = z \Rightarrow \boxed{x = e^z}$

(2) ~~$xD = \theta, x^2D^2, x^3D^3 =$~~

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$, $x^3D^3 = \theta(\theta-1)(\theta-2)$ ...

(3) Substitute in (1); Reduce to linear differential equation.

(4) Answer must be in terms of x.

* imp Pattern Quest

Q1. Solve $x^2y'' - 2xy' + 2y = \frac{1}{x}$

Sol. Given $x^2y'' - 2xy' + 2y = \frac{1}{x}$

$\Rightarrow x^2D_y^2 - 2xD_y + 2y = x^{-1}$

$\Rightarrow [x^2D^2 - 2xD + 2]\, y = x^{-1}$ ——— (1)

$\log_e x = z \Rightarrow x = e^z$

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$, sub in (1)

$[\theta(\theta-1) - 2\theta + 2]y = (e^z)^{-1}$

$[\theta^2 - \theta - 2\theta + 2]y = e^{-z}$

$[\theta^2 - 3\theta + 2]\, y = e^{-z} \longrightarrow$ type I.

Auxillary eq^n : $m^2 - 3m + 2 = 0$

$m = 1, 2$

C.F $\Rightarrow y_c = c_1e^{1z} + c_2e^{2z}$

P.I $\Rightarrow y_p = \frac{1}{D^2 - 3D + 2} e^{-z}$

Replace $\boxed{D = a = -1}$

$y_p = \frac{1}{(-1)^2 - 3(-1) + 2} e^{-z}$

$= \frac{1}{1+3+2} e^{-z}$

$= \frac{1}{6} e^{-z} = \frac{e^{-z}}{6}$

General solution $y = y_c + y_p$

$y = c_1e^z + c_2e^{2z} + \frac{e^{z}}{6}$

$y = c_1x + c_2x^2 + \frac{1}{6x}$

$\because e^z = x$

$e^{-z} = x^{-1} = \frac{1}{x}$

$e^{2z} = x^2$.

This $[-y'']$

Q2. Solve $x^2y'' - xy' + y = x^2$.

Sol: Given equation $x^2y'' - xy' + y = x^2$

$\Rightarrow [x^2D^2 - xD + 1]y = x^2$ —— ①.

$\log_e^x = z \Rightarrow x = e^z$.

$xD = \theta$, $x^2D^2 = \theta(\theta-1)$ substitute in ①

$\Rightarrow [\theta(\theta-1) - \theta + 1]y = (e^z)^2$

$\Rightarrow [\theta^2 - \theta - \theta + 1]\ y = e^{2z}$

$\Rightarrow [\theta^2 - 2\theta + 1]\ y = e^{2z}$.

Auxillary eq^n $m^2 - 2m + 1 = 0$

$m = 1, 1$.

C.F $\Rightarrow y_c = [c_1 + c_2 z]e^z$.

P.I $\Rightarrow y_p = \dfrac{1}{(D^2 - 2D + 1)} e^{2z}$ $\boxed{D = a = 2}$.

$= \dfrac{1}{4-4+1} e^{2z}$

$= e^{2z}$

General Solution $y = y_c + y_p$

$y = [c_1 + c_2 z]e^z + e^{2z}$

$= [c_1 + c_2 \log_e^x]\,x + x^2$ //

$e^z = x$

$z = \log_e^x$

$e^{2z} = x^2$

Q. Solve $x^2y''+2xy'=\cos[\log x]$.

Sol. Given equation $x^2y''+2xy'=\cos[\log x]$

$\Rightarrow [x^2D^2+2xD]y=\cos[\log x]$ —— ①

$\log_e x = z \Rightarrow x=e^z$.

$xD=\theta,\ x^2D^2=\theta(\theta-1)$ Substitute in ①

$\Rightarrow [\theta(\theta-1)+2\theta]y=\cos[z]$

$\Rightarrow [\theta^2-\theta+2\theta]y = \cos z$

$\Rightarrow [\theta^2+\theta]y = \cos z$

Auxillary equation ; $m^2+m=0$

$m=0,-1$

$C.F \Rightarrow y_c = c_1e^{0x}+c_2e^{-1z} = c_1+c_2e^{-z}$

$P.I \Rightarrow y_p = \frac{1}{(D^2+D)}\cos z$ $\boxed{D^2=-a^2=-1}$

$= \frac{1}{-1+D}\cos z$

$= \frac{1}{(D-1)(D+1)}(D+1)\cos z$

$= \frac{(D+1)\cos z}{D^2-1}$

$= \frac{D\cos z+\cos z}{-1-1}$ $D\cos z = -\sin z$.

$= \frac{-\sin z+\cos z}{-2}$

$y_p = \frac{\cos z-\sin z}{2}$

$\therefore$ General Solution : $y=y_c+y_p = c_1+c_2e^{-z}+\frac{\cos z-\sin z}{2}$

$= c_1+c_2x^{-1}+\frac{\cos\log x-\sin\log x}{2}$

HW

Q4. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$.

Sol. Given equation $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$

$\Rightarrow (x^2D^2 - xD + 1)y = 2\log x$ — ①

$\log_e x = z \Rightarrow x = e^z$, $xD = \theta$, $x^2D^2 = \theta(\theta-1)$ sub in ①.

$\Rightarrow [\theta(\theta-1) - \theta + 1]y = 2(z)$.

$\Rightarrow (\theta^2 - \theta - \theta + 1)y = 2z$

$\Rightarrow (\theta^2 - 2\theta + 1)y = 2z$

$(m-1)^2 = 0$
$m = 1, 1$.

Auxillary eqn: $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

C.F $\Rightarrow y_c = [c_1 + c_2 z]e^z$

P.I $\Rightarrow y_p = \frac{1}{D^2 - 2D + 1} 2z$

$= \frac{1}{[1 + (D^2 - 2D)]} 2z$

$= 2[1 + (D^2 - 2D)]^{-1} z$

$= 2[1 - (D^2 - 2D)] z$

$= 2[1 + 2D] z$

$= 2\left[z + \frac{2}{x}\right]$

$= 2\left[\log_e x + \frac{2}{x}\right]$ //.

$2(1+2\theta)z$,

$= 2z + 4\theta z$

$= 2\log x + 4\theta \log x$,

$= 2\log x + 4D\log x$,

$= 2\log x + \frac{4}{x}$,

$= 2\log x + \frac{4}{x}$

$= 2\left(\log x + \frac{2}{x}\right)$.

③ Variation of Parameters [VOP] 08|08|25.

(7 Marks)

An equation of the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x)$ be a second degree differential equation.

Working Rule :

① Find $y_c = c_1 u(x) + c_2 v(x)$

② Write $y_p = A u(x) + B v(x)$

③ Find Wronskian $W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

④ Find the parameters

$A = -\int \frac{v}{W} R\,dx$ and $B = \int \frac{uR}{W} dx$.

⑤ General Solution $y = y_c + y_p$.

V.imp

Q1. Solve $(D^2+1)y = \sec x$ by variation of parameters.

Sol Given $(D^2+1)y = \sec x$

Auxillary eq^n $m^2+1=0$

$m = 0 \pm i$

C.F $\Rightarrow y_c = e^{0x}[c_1 \cos 1x + c_2 \sin 1x] = [c_1\cos x + c_2 \sin x]$.

P.I $\Rightarrow y_p = A\cos x + B\sin x$ —— ①.

$u(x) = \cos x$, $v(x) = \sin x$

Wronskian $W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$

$\int \frac{f'(x)}{f(x)} = \log|f(x)| + C$

$A = -\int \frac{vR}{W} dx = -\int \frac{\sin x \cdot \sec x}{1} dx = -\int \frac{\sin x}{\cos x} dx = \log|\cos x|$.

$B = \int \frac{uR}{W} dx = \int \frac{\cos x \cdot \sec x\, dx}{1} = x$ → Substitute in ①

$y_p = \log\cos x + x\sin x$.

$\therefore$ General solution $= y_c + y_p$

$= c_1\cos x + c_2 \sin x + \log\cos x + x\sin x$

Q2. Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$ by method of variation of parameters.

Sol: Given $\frac{d^2y}{dx^2} + 4y = \tan 2x \Rightarrow (D^2+4)y = \tan 2x$

Auxillary equation : $m^2+4=0$

$m^2 = -4$

$m = 0 \pm 2i$

C.F $\Rightarrow y_c = c_1\cos 2x + c_2 \sin 2x$

P.I $\Rightarrow y_p = A\cos 2x + B\sin 2x$

$u(x) = \cos 2x$, $v(x) = \sin 2x$

Wronskian $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2(\cos^2 2x + \sin^2 2x) = 2(1) = 2.$

$A = -\int \frac{vR}{w} dx = -\int \frac{\sin 2x \cdot \tan 2x}{2} dx = -\frac{1}{2}\int \sin 2x \cdot \frac{\sin 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int \frac{(1-\cos^2 2x)}{\cos 2x} dx$

$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx$

$= \frac{+1}{2}\int (\cos 2x - \sec 2x) dx$

$A = \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\log(\sec 2x + \tan 2x)}{2}\right]$

$B = \int \frac{uR}{w} dx = \int \frac{\cos 2x \cdot \tan 2x}{2} dx = \int \frac{\cancel{\cos 2x}}{2} \cdot \frac{\sin 2x}{\cancel{\cos 2x}} dx$

$= \int \frac{\sin 2x}{2} = \frac{-\cos 2x}{4}$

Substitute in ① we get;

$y_p = A\cos 2x + B\sin 2x$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\log(\sec 2x \tan 2x)}{2}\right]\cos 2x + \left(-\frac{\cos 2x}{4}\right)\sin 2x$

$= \left[\frac{\sin 2x}{4} - \log\frac{\sec 2x \tan 2x}{4}\right]\cos 2x - \frac{\cos 2x}{4}\cdot\sin 2x$

General solution $y = y_c + y_p$

$y = C_1\cos 2x + C_2\sin 2x + \left[\frac{\sin 2x}{4} - \log\frac{\sec 2x \tan 2x}{4}\right]\cos 2x$

$- \frac{\cos 2x}{4}\cdot\sin 2x.$

Q3. Solve $(D^2+1)y = \operatorname{cosec} x \cot x$ by variation of parameters.

Sol: Given $(D^2+1)y = \operatorname{cosec} x \cot x$

Auxillary equation : $m^2+1=0$

$m^2=-1$

$m = 0 \pm i$

C.F $\Rightarrow y_c = e^{0x}[c_1\cos 1x + c_2\sin 1x]$

$y_c = [c_1\cos x + c_2\sin x]$ $u = \cos x$

P.I $\Rightarrow y_p = A\cos x + B\sin x$ — ① $v = \sin x$

Wronskian $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$

$= \cos^2 x + \sin^2 x = 1$

$A = -\int \frac{vR}{w}dx = -\int \frac{\sin x\cdot \operatorname{cosec} x \cot x\, dx}{1} = -\int \cot x\, dx$

$= -\log|\sin x| + c$

$= \log(\sin^{-1} x)$

$= \log \operatorname{cosec} x$

$B = \int \frac{uR}{w}dx = \int \frac{\cos x\cdot \operatorname{cosec} x \cdot \cot x}{1}$

$\int \frac{\cos x\cdot \frac{1}{\sin x}\cdot \cot x}{1} = \int \frac{\cot x\cdot \cot x}{1} = \int \cot^2 x$

$\int \cot^2 x \, dx$

$\Rightarrow \int (\operatorname{cosec}^2 x - 1) dx$

$\Rightarrow -\cot x - x + c$.

Substitute in ①

$y_P = A\cos x + B\sin x$.

$y_P = \log\operatorname{cosec} x \cdot \cos x + (-\cot x - x)\sin x$.

$y_P = \log\operatorname{cosec} x \cos x - (\cot x + x)\sin x$

$\therefore$ General Solution is $y = y_c + y_P$

$y = c_1\cos x + c_2\sin x + \log\operatorname{cosec} x \cdot \cos x - (\cot x + x)\sin x$.

Q4. Solve $(D^2 - 2D)y = e^x \sin x$ by variation of parameters.

Sol. Given $(D^2 - 2D)y = e^x \sin x$.

Auxillary equation : $m^2 - 2m = 0$

$m(m-2) = 0$

$m = 0, m - 2 = 0$

$m = 0, m = 2$.

C.F $\Rightarrow y_c = c_1 e^{0x} + c_2 e^{2x} = c_1(1) + c_2 e^{2x} = c_1 + c_2 e^{2x}$.

P.I $\Rightarrow y_P = Ae^{0x} + Be^{2x}$ ① $\Rightarrow$ where $u(x) = 1$, $v(x) = e^{2x}$.

Wronskian $\omega = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} - 0 = 2e^{2x}$.

$A = -\int \frac{VR}{\omega} dx = -\int \frac{\cancel{e^{2x}} \cdot e^x \sin x}{2\cancel{e^{2x}}} = -\int \frac{e^x \sin x}{2} dx$

Formula : $\boxed{\int e^{ax} \sin x \, dx = \frac{e^{ax}}{a^2 + b^2}[a\sin bx - b\cos bx]}$

$-\frac{1}{2}\int e^{x}\sin x\,dx$

$\Rightarrow -\frac{1}{2}\left[\frac{e^{1x}}{1+1}[\sin x-\cos x]\right]$

$A \Rightarrow -\frac{e^{x}}{4}[\sin x-\cos x]$

$B=\int \frac{uR}{W}dx = \int \frac{1\cdot e^{x}\sin x\,dx}{2e^{2x}} = \frac{1}{2}\int e^{-x}\sin x\,dx = \frac{1}{2}\int \frac{e^{-x}}{1+1}[-\sin x-\cos x]$

$= \frac{e^{-x}}{4}[-\sin x-\cos x]$

Substitute A and B in equation ① we get:

$y_P = -\frac{e^{-x}}{4}[\sin x-\cos x] - \left[-\frac{e^{-x}}{4}[-\sin x+\cos x]\right]e^{2x}$

$= -\frac{e^{x}}{4}\sin x + \frac{e^{-x}}{4}\cos x + \left[\frac{e^{x}}{4}\sin x - \frac{e^{x}}{4}\cos x\right]e^{2x}$

$= -\frac{e^{x}}{4}\sin x + \cancel{\frac{e^{-x}}{4}\cos x} - \cancel{\frac{e^{x}}{4}\cos x} - \frac{e^{x}}{4}\sin x \cdot e^{2x}$

$= -\frac{e^{x}\sin x}{2}$

$\therefore$ General solution is $y = y_c + y_P$.

$y = c_1 + c_2 e^{2x} - \frac{e^{x}\sin x}{2}$

Q50 Solve $y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$

Sol. Given $(D^2-6D+9)y = \frac{e^{3x}}{x^2}$

Auxillary equation: $m^2-6m+9=0$

$m=3,3$

$C.F \Rightarrow y_c = [c_1 + c_2 x] e^{3x}$

$y_c = c_1 e^{3x} + c_2 x e^{3x}$.

$P.I \Rightarrow y_p = A e^{3x} + B x e^{3x}$ where $u(x) = e^{3x}$, $v(x) = x e^{3x}$

Wronskian $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & x e^{3x} + e^{3x} \end{vmatrix}$

$= 3x e^{6x} + e^{6x} - 3x e^{6x}$

$= e^{6x}$

$A = -\int \frac{vR}{w} dx = -\int \frac{x e^{3x} \cdot \frac{e^{3x}}{x^2}}{e^{6x}} dx = -\int \frac{1}{x} dx = -\log x = \log \frac{1}{x}$

$= \log x^{-1}$

$= \log \frac{1}{x}$.

$B = \int \frac{uR}{w} dx = \int \frac{e^{3x} \cdot e^{3x}}{e^{6x} \cdot x^2} dx = \int x^{-2} dx = \frac{x^{-1}}{-1} = -\frac{1}{x}$.

$y_p = \log \frac{1}{x} \cdot e^{3x} - \frac{1}{\cancel{x}} \cancel{x} e^{3x}$

$y_p = \log \frac{1}{x} \cdot e^{3x} - e^{3x}$.

$\therefore$ General solution $y = y_c + y_p$

$y = c_1 e^{3x} + c_2 x e^{3x} + \log \frac{1}{x} e^{3x} - e^{3x}$ //.

H/w: Q. $(D^2+1)y = x$

Q. $y'' - 2y' + y = e^x \log x$.

Q. Solve $(D^2+1)y = x$.

Sol: Given $(D^2+1)y = x$

Auxillary equation $m^2+1=0$

$m^2 = -1$

$m = 0 \pm i$

$C.F \Rightarrow y_c = e^{0x}[c_1\cos 1x + c_2 \sin 1x] = c_1\cos x + c_2\sin x$.

$P.I \Rightarrow y_p = A\cos x + B\sin x$ where $u(x) = \cos x$ $v(x) = \sin x$.

wronskian $w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$.

$A = -\int \frac{vR}{w} dx = -\int \frac{\sin x \cdot x}{1} = -\int x\sin x =$

	d	i
+	x	$\sin x$
−	1	$-\cos x$
+	0	$\mp\sin x$

$-\int -x\cos x + \sin x$.

$\int f(x)g(x) = f(x)\int g(x) - \int \frac{d}{dx} f(x) \int g(x)dx \, dx$.

xe^x

$xe^x - e^x$

xe^{-x}

$1.e^{-x} + x.e^{-x}(-1)$

$e^{x} - xe^x$

uv

$u'v + uv'$

(4) Method of Reduction of Linear Differential Equation.

An equation of the form $a_0(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_2(x)y = 0$ & $y_1(x)$ is known as non trivial solution of the differential equation of second degree.

→ Working Rule:

1. Find $P(x) = \frac{a_1(x)}{a_0(x)}$
2. Find $v(x) = \frac{1}{y^2} e^{-\int P dx}$
3. Find $u(x) = \int v(x) dx$
4. Find $y_2(x) = u(x) y_1(x)$
5. General Solution $y = Ay_1(x) + By_2(x)$.

Q1. If $y_1 = \frac{1}{x}$ is one of the solution of the differential equation $x^2y'' + 4xy' + 2y = 0$ then find the general solution of differential equation by reducing its order.

Sol. Given $x^2y'' + 4xy' + 2y = 0$, $y_1 = \frac{1}{x}$.

$a_0(x) = x^2$. $a_1(x) = 4x$, $a_2(x) = 2$.

$$P(x) = \frac{a_1(x)}{a_0(x)} = \frac{4x}{x^2} = \frac{4}{x}$$

$$v(x) = \frac{1}{\left(\frac{1}{x}\right)^2} e^{-\int \frac{4}{x} dx}$$

$$= x^2 e^{-4\log x} dx$$

$$= x^2 e^{\log x^{-4}}$$

$$= x^2 \cdot x^{-4} = x^{-2} = \frac{1}{x^2}$$

$u(x) = \int v(x)dx = \int x^{-2} dx = \frac{x^{-1}}{-1} = -\frac{1}{x}$ ✓

$y_2(x) = u(x)\, y(x) = -\frac{1}{x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$\therefore$ General Solution is $y = y_c + y_p$ $\quad y = A y_1(x) + B y_2(x)$

$$y = A \cdot \frac{1}{x} + B \cdot \left(\frac{-1}{x^2}\right)$$

$$y = A \cdot \frac{1}{x} - B \cdot \frac{1}{x^2} \text{ //.}$$

Q2. If $y_1 = e^x$ is one of the solution of the differential equation $y'' + 3y' + 4y = 0$ then find the general solution of differential equation by reducing its order.

Sol: Given $y'' + 3y' + 4y = 0$

$y_1 = e^x$. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 4y = 0$

$a_0(x) = 1$, $a_1(x) = 3$, $a_2(x) = 4$

$P(x) = \frac{a_1(x)}{a_0(x)} = \frac{3}{1} = 3$

$v(x) = \frac{1}{(e^x)^2} \cdot e^{-\int 3\,dx} = \frac{1}{e^{2x}} e^{-3\int dx} = \frac{1}{e^{2x}} e^{-3x} = e^{-5x}$

$u(x) = \int v(x) dx = \int e^{-5x} = \frac{e^{-4x}}{-4}$

$y_2(x) = u(x) \cdot y_1(x) = \cancel{\frac{e^{-5x}}{-5}} \times e^x \times \frac{e^{-4x}}{-4} = \frac{e^{-4x+x}}{-4} = \frac{e^{-3x}}{-4}$

$\therefore$ General Solution $y = Ae^x - B\frac{e^{-3x}}{-4}$

Q3- Find the orthogonal trajectories of the family of circles passing through $(0,2),(0,-2)$.

Sol. Let equation of circle be $x^2+y^2+2gx+2fy+c=0 \longrightarrow (1)$ circle passes through $(0,2),(0,-2)$ then.

$(0,2) \Rightarrow 0+4+2g(0)+4f+c=0 \Rightarrow 4+4f+c=0$

$(0,-2) \Rightarrow 0+4+2g(0)-4f+c=0 \Rightarrow 4-4f+c=0$

$8 \qquad 2c=0 \qquad \boxed{c=-4}$

$8f=0 \qquad \boxed{f=0}$

Sub in (1).

then $\boxed{x^2+y^2+2gx-4=0 \longrightarrow (2)}$.

D.w.r.t x.

$2x+2y\frac{dy}{dx}+2g=0$

$g=-x-y\frac{dy}{dx}$ Substitute in (2).

$\Rightarrow x^2+y^2+2x\left(-x-y)\frac{dy}{dx}\right)-4=0$

$\Rightarrow x^2+y^2-2x^2-2xy\frac{dy}{dx}-4=0$

$\Rightarrow y^2-x^2-2xy\frac{dy}{dx}-4=0$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$y^2-x^2+2xy\frac{dx}{dy}=4$

$y^2_{dy}-x^2dy+2xydx=4dy \quad \div y^2 \Rightarrow dy+\frac{y2xdx-x^2dx}{y^2}=4y^{-2}dy$

Integration on both sides;

$\Rightarrow \int dy+\int d\left(\frac{x^2}{y}\right)=\int 4y^{-2}dy+c$

$y+x^2/y=-4/y+c \quad \Rightarrow \boxed{y^2+x^2-cy-4=0}$ is Required orthogonal trajectory.

11/07/2024. Initial Value Problems: [Find the values of constants]

(5)

18. $y''' - 2y'' - 5y' + 6y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$.

Sol: Given $y''' - 2y'' - 5y' + 6y = 0$ — (I).

(1) can be written as

$(D^3 - 2D^2 - 5D + 6)y = 0$.

Auxillary eqn: $m^3 - 2m^2 - 5m + 6 = 0$

$m = 1, -2, 3$.

C.F => $y_c = c_1 e^{1x} + c_2 e^{-2x} + c_3 e^{3x}$.

$$\boxed{y(x) = c_1 e^{x} + c_2 e^{-2x} + c_3 e^{3x}}$$

$y(0) = c_1 e^0 + c_2 e^0 + c_3 e^0 = 0$.

$\Rightarrow c_1 + c_2 + c_3 = 0$ — (1).

$y'(x) = c_1 e^{x} + c_2 e^{-2x}(-2) + c_3 e^{3x}(3)$

$y'(0) = c_1 e^0 - 2c_2 e^0 + 3c_3 e^0 = 0$

$c_1 - 2c_2 + 3c_3 = 0$ — (2).

$y''(x) = c_1 e^{x} + c_2 e^{-2x}(-2)^2 + c_3 e^{3x}(3)^2$

$y''(0) = c_1 e^0 + c_2 e^0 (4) + c_3 e^0 (9) = 1$

$= c_1 + 4c_2 + 9c_3 = 1$ — (3)

Solve (1), (2), (3) in (I).

$\Rightarrow c_1 = \frac{-1}{6}$, $c_2 = \frac{1}{15}$, $c_3 = \frac{1}{10}$

Sub in $y(x) = -\frac{1}{6} e^{x} + \frac{1}{15} e^{-2x} + \frac{1}{10} e^{3x}$

$y(0)=-1,\ y'(0)=2$ — ①

Q2. Solve $4y''+12y'+9y=0$, $y(0)=-1$, $y'(0)=2$

Sol. Given $4y''+12y'+9y=0$, $y(0)=-1$, $y'(0)=2$ — ①

① can be written as $(4D^2+12D+9)y=0$

Auxillary equation : $4m^2+12m+9=0$

$$m=-\frac{3}{2}, -\frac{3}{2}$$

$$y_c=(c_1+c_2x)e^{-3/2x}$$

$$\boxed{y(x)=(c_1+c_2x)e^{-3/2x}}$$

$y(0)=-1 \Rightarrow x=0, y=-1$

$$\Rightarrow y(0)=(c_1+c_2(0))e^{-3/2\,0} \Rightarrow (c_1+0)e^{0} = \boxed{c_1=-1}$$ — ①

$$y'(x)=\cancel{(0+c_2)e^{-3/2x}}\ d'\left[c_1e^{-3/2x}+c_2xe^{-3/2x}\right]$$

$$=c_1e^{-3/2x}+c_2\left(1e^{-3/2x}+xe^{-3/2x}\left(\frac{-3}{2}\right)\right)$$

$$=c_1e^{-3/2x}+c_2e^{-3/2x}-\frac{3}{2}xe^{-3/2x}$$

$y'(0)=2$

$x=0, y=2 \Rightarrow c_1e^{-3/2(0)}+c_2e^{-3/2(0)}-\frac{3}{2}(0)e^{-3/2(0)}$

$\Rightarrow c_1+c_2=2$ — ②

from ① $c_1=-1 \Rightarrow -1+c_2=2$

$c_2=2+1=3$

$c_2=3$ — ③

Substitute ①, ③ in $y(x)$

we get ; $\boxed{(-1+3x)e^{-3/2x}=y(x)}$