03/05/2025. # Unit - I Differential Equations of I Order.

2 Marks:

(1.) Exact Differential equation.

An equation of the form $Mdx + Ndy = 0$

→ check Exactness i.e $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

→ Its General Solution,

$$\int_{y=\text{constant}} M dx + \int N [\text{Terms free from } x]\, dy = C.$$

1Q. Solve $[2x-y+1]dx + [2y-x-1]dy = 0$ — (1)

Sol: $M = 2x-y+1$

$N = 2y-x-1$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ → (1) is Exact differential equation.

General Solution is:

$$\int_{y=\text{constant}} [2x-y+1]dx + \int [2y-1]dy = C$$

$$\Rightarrow \cancel{2}\frac{x^2}{\cancel{2}} - yx + x + \cancel{2}\frac{y^2}{\cancel{2}} - y = C$$

$$\Rightarrow \boxed{x^2+y^2-xy+x-y=C}$$

2Q. Solve $[y^2-2xy]dx - [x^2-2xy]dy = 0$

Sol: Given $[y^2-2xy]dx - [x^2-2xy]dy = 0$ — (1)

$Mdx + Ndy \Rightarrow M = y^2-2xy$, $N = -x^2+2xy$

$\frac{\partial M}{\partial y} = 2y-2x$ $\frac{\partial N}{\partial x} = -2x+2y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$ ① is Exact differential equation.

General Solution is :

$$\int_{y=\text{constant}} M\,dx + \int N[\text{Terms free from } x]\,dy = C$$

$$\int_{y=\text{constant}} [y^2 - 2xy]\,dx + \int \underbrace{\cancel{-x^2 + 2xy}}_{\text{free from } x}{}^{0} [dy] = C$$

$\Rightarrow \int y^2 dx - 2xy\,dx + \int 0\,dy = C$

$\Rightarrow y^2 x - \cancel{2}\frac{x^2}{\cancel{2}} y = C$

$\Rightarrow y^2 x - x^2 y = C$

$\Rightarrow xy[y - x] = C$,

3 Q. Solve $\left[y\left[1 + \frac{1}{x}\right] + \cos y\right]dx + \left[x + \log x - x\sin y\right]dy = 0$ — ①

Sol. Given $\left[y\left[1 + \frac{1}{x}\right] + \cos y\right]dx + \left[x + \log x - x\sin y\right]dy = 0$

$\Rightarrow \left[y + \frac{y}{x} + \cos y\right]dx + \left[x + \log x - x\sin y\right.$

where $M = y + \frac{y}{x} + \cos y$, $N = x + \log x - x\sin y$

$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$, $\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - 1\cdot\sin y$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \longrightarrow$ ① is Exact differential equation,

General Solution is :

$$\int_{y=\text{constant}} M\,dx + \int N[\text{Terms free from } x]\,dy = C.$$

$$\int \left[y + \frac{y}{x} + \cos y\right] dx + \int 0\, dy = c$$

y=constant

$$\int y\, dx + \frac{y}{x}\, dx + \cos y\, dx = c$$

$$\Rightarrow \boxed{yx + y\log x + x\cos y = c}$$

4Q. Solve $\left[1+\frac{e^{x/y}}{\cancel{x}}\right]dx + \left[1-\frac{x}{y}\right]e^{x/y}\, dy = 0$

Sol: Given $\left[1+e^{x/y}\right]dx + \left[1-\frac{x}{y}\right]e^{x/y}\, dy = 0$

where $M = 1+e^{x/y}$

$N = \left[1-\frac{x}{y}\right]e^{x/y}$

$$\frac{\partial M}{\partial y} = 0 + e^{x \cdot \log y}$$

$$\frac{\partial N}{\partial x} = e^{x/y} - \frac{x e^{x/y}}{y}$$

$(uv)' = u'v + uv'$

$$= e^{1/y} - \left[\frac{1}{y} e^{x/y} + \frac{x}{y} e^{1/y}\right]$$

$$= e^{1/y} - \left[\frac{e^{x/y} + x e^{1/y}}{y}\right]$$

$$= e^{1/y} - \left[\frac{e^{x} \cdot e^{1/y} + x \cdot e^{1/y}}{y}\right]$$

$$= e^{1/y} - \left[\frac{e^{1/y}\left[e^{x} + x\right]}{y}\right]$$

Given $[1+e^{x/y}]dx+[1-\frac{x}{y}]e^{x/y}dy=0$ —— ①

where $M=1+e^{x/y}$ $N=[1-\frac{x}{y}]e^{x/y}$

$\frac{\partial m}{\partial y}=0+e^{x/y}\cdot\left(-\frac{x}{y^2}\right)$, $\frac{\partial N}{\partial x}=e^{x/y}\cancel{/}\frac{1}{y}-\frac{x}{y}\cdot e^{x/y}\frac{1}{y}-\cancel{\frac{1}{y}}e^{x/y}$

$=e^{x/y}\left(-\frac{x}{y^2}\right)$

$\frac{\partial m}{\partial y}=\frac{\partial N}{\partial x}$

$\therefore$ ① is Exact differential equation.

General Solution is

$\int\limits_{y=constant} Mdx+\int Ndy\,[\text{Terms free from } x]=c$

$\int\limits_{y=constant}[1+e^{x/y}]dx+\int 0\,dy=c$

$\Rightarrow \int 1\,dx+e^{x/y}dx=c$

$\Rightarrow x+\frac{e^{x/y}}{\frac{1}{y}}=c$

$\Rightarrow \boxed{x+ye^{x/y}=c}$

5Q. Find the value of 'a' and 'b' so that the differential equation is exact:

$(x^2+axy-2y^2)dx+(y^2-4xy+bx^2)dy=0$.

Sol. Given equation $(x^2+axy-2y^2)dx+(y^2-4xy+bx^2)dy=0$ —

where $M=x^2+axy-2y^2$, $N=y^2-4xy+bx^2$

$\frac{\partial m}{\partial y}=ax-2\cdot 2y$, $\frac{\partial N}{\partial x}=-4y+2bx$

$ax-4y$

Since the given differential equation is exact

$\Rightarrow \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$

$\Rightarrow ax - \cancel{4y} = \cancel{-4y} + 2bx$

$\Rightarrow a\cancel{x} = 2b\cancel{x}$

$\Rightarrow \frac{a}{b} = \frac{2}{1}$

$\Rightarrow \boxed{a=2, \; b=1}$.

② Non Exact Differential Equation.

Type ①:

① $d(xy) = xdy + ydx$

② $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

③ $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{xdy - ydx}{x^2+y^2}$

④ $d\left[\log(xy)\right] = \frac{xdy + ydx}{xy}$

$\leadsto d\left[\left(\frac{1}{2}\right)\log (x^2+y^2)\right] = \frac{xdy + ydx}{x^2+y^2}$

$\leadsto d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right] = \frac{xdy - ydx}{x^2-y^2}$

$\leadsto d\left[\log\left(\frac{y}{x}\right)\right] = \frac{xdy - ydx}{xy}$

Solve $[x^4e^x - 2mxy^2]dx + 2mx^2y\,dy = 0$

Given $[x^4e^x - 2mxy^2]dx + 2mx^2y\,dy = 0$ — ①

where $M = x^4e^x - 2mxy^2$, $N = 2mx^2y$

$\frac{\partial M}{\partial y} = -4mxy$, $\frac{\partial N}{\partial x} = 4mxy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$,

$\therefore$ It is not exact differential equation.

$x^4 e^x dx - 2mxy^2 dx + 2mx^2 y\, dy = 0$

$x^4 e^x dx - 2m[xy^2 dx - x^2 y\, dy] = 0$

$\div x^4 \Rightarrow e^x dx + \dfrac{m[x^2 2y\, dy - y^2 2x\, dx]}{x^4} = 0$

$\Rightarrow e^x dx + m\left[\dfrac{x^2 2y\, dy - y^2 2x\, dx}{(x^2)^2}\right] = 0$

$e^x dx + m\, d\left(\dfrac{y^2}{x^2}\right) = 0$

Integrate.

$\int e^x dx + m \int d\left(\dfrac{y^2}{x^2}\right) = c$

$e^x + m\dfrac{y^2}{x^2} = c$ //.

2Q. Solve $y[2xy + e^x]dx + e^x dy = 0$.

Sol: Given $y[2xy + e^x]dx + e^x dy = 0$ —— ①

$[2xy^2 + e^x y]\, dx + e^x dy = 0$

where $M = 2xy^2 + ye^x$, $N = e^x$

$\dfrac{\partial M}{\partial y} = 4xy + e^x$, $\dfrac{\partial N}{\partial x} = e^x$

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

$\therefore$ It is not exact differential equation.

$2xy^2 dx + ye^x dx + e^x dy = 0$

$\div y^2 \Rightarrow 2x\,dx + \dfrac{y\, e^x dx + e^x dy}{y^2} = 0$

$2x\,dx + d\left(\dfrac{e^x}{y}\right) = 0$

Integrate $\Longrightarrow \int 2x\,dx + \int d\left(\dfrac{e^x}{y}\right) = c \Rightarrow \dfrac{\cancel{2}x^2}{\cancel{2}} + \dfrac{e^x}{y}$

$\Rightarrow x^2 + e^x/y = C$

Solve $y\,dx - x\,dy = a[x^2+y^2]\,dx$.

$y\,dx - x\,dy = a[x^2+y^2]\,dx \longrightarrow ①$

① is not exact differential equation.

$\div x^2+y^2 \Rightarrow \dfrac{y\,dx - x\,dy}{x^2+y^2} = \dfrac{a\,\cancel{[x^2+y^2]}\,dx}{\cancel{[x^2+y^2]}}$

$\Rightarrow \dfrac{-[x\,dy - y\,dx]}{x^2+y^2} = a\,dx$

$\Rightarrow \dfrac{x\,dy - y\,dx}{x^2+y^2} = -a\,dx$

~~Integra~~

$\Rightarrow d\left(\tan^{-1}\frac{y}{x}\right)$

Integrate.

$\int d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = \int -a\,dx$

$\tan^{-1}\frac{y}{x} = -ax$

$\boxed{\tan^{-1}\frac{y}{x} + ax = c}$ //

Solve $y\,dx - x\,dy - 3x^2y^2e^{x^3}dx = 0$ — ① $\therefore$ it is not exact differential equation.

$Mdx+Ndy$ is not exact so $\therefore$

$y\,dx - x\,dy = 3x^2y^2e^{x^3}dx$

$\dfrac{y\,dx - x\,dy}{y^2} = \dfrac{3x^2\cancel{y^2}e^{x^3}dx}{\cancel{y^2}}$

$\Rightarrow d\left(\frac{x}{y}\right) = d(e^{x^3})$

Integrate;

$\int d\left(\frac{x}{y}\right) = \int d(e^{x^3})$

$\Rightarrow \boxed{\frac{x}{y} = e^{x^3} + c}$

③ **Integrating factor:**

If the given differential equation is not exact; then it may be transformed into exact differential equation by multiplying it by a term is called as integrating factor.

Type-VI: [Homogeneous Differential equation]

* If the given non exact differential equation is homogeneous $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is integrating factor.

1Q. Solve $(3xy - 2y^2)dx + (x^2 + 2xy)dy = 0$ — ①

Sol. Given $(3xy-2y^2)dx + (x^2+2xy)dy = 0$

where $M = 3xy - 2y^2$, $N = x^2 + 2xy$

$\frac{\partial M}{\partial y} = 3x - 4y$, $\frac{\partial N}{\partial x} = 2x + 2y \Rightarrow \therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ ① Not exact differentia

$Mx + Ny = (3xy - 2y^2)x + (x^2 + 2xy)y$

$= 3x^2y - \cancel{2xy^2} + x^2y + \cancel{2xy^2}$

$= 4x^2y.$

$$I.F = \frac{1}{Mx+Ny} = \frac{1}{4x^2y}$$

$$① \times IF = \frac{1}{4x^2y}\left[3xy-2y^2\right]dx + \frac{1}{4x^2y}\left[x^2+2xy\right]dy = 0$$

$$\left[\frac{3}{4x} - \frac{y}{2x^2}\right]dx + \left[\frac{1}{4y} + \frac{2}{4x}\right]dy = 0$$

This is exact differential equation.

$\Rightarrow$ General Solution : $\int M dx + \int N\,[\text{Terms free from } x]\,dy = c$

$$\int \left[\frac{3}{4x} - \frac{y}{2}x^{-2}\right]dx + \int \frac{1}{4y}\,dy = c$$

$$\Rightarrow \frac{3}{4}\log x - \frac{y}{2}\frac{x^{-1}}{-1} + \frac{1}{4}\log y = c$$

$$\Rightarrow \frac{3\log x}{4} + \frac{y}{2x} + \frac{1}{4}\log y = c$$

$$\Rightarrow 3\log x + \frac{2y}{x} + \log y = c \qquad [\because \log x^3 = 3\log x]$$

$$\Rightarrow \frac{2y}{x} + \log x^3 + \log y = c \qquad [\because \log x^3 + \log y = \log x^3y]$$

$$\Rightarrow \frac{2y}{x} + \log x^3y = c$$

2Q. Solve $(x^3+y^3)dx - xy^2dy = 0$.

Sol. Given $(x^3+y^3)\,dx - xy^2dy = 0$ —— ①

where $M = x^3+y^3$; $N = -xy^2$

$$\frac{\partial M}{\partial y} = 3y^2 , \quad \frac{\partial N}{\partial x} = -y^2$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ It is not Exact Differential equation.

Given differential equation ① is Homogeneous Differential equation.

$\Rightarrow Mx + Ny = (x^3+y^3)x + (-xy^2)y$

$\Rightarrow x^4 + \cancel{xy^3} - \cancel{xy^3} = x^4$

$I.F = \dfrac{1}{Mx+Ny} = \dfrac{1}{x^4}$

① $\times$ I.F $= \dfrac{1}{x^4}[x^3+y^3]dx + \dfrac{1}{x^4}[-xy^2dy] = 0$

$= \left[\dfrac{1}{x} + \dfrac{y^3}{x^4}\right]dx - \dfrac{y^2}{x^3}dy = 0$

This is an exact differential equation.

$\Rightarrow$ General Solution: $\int Mdx + \int N[\text{Terms free from } x]dy = c$

$\cancel{\int x^3 + y^3 dx + \int 0 dy = c}$

$\Rightarrow \cancel{\left[\dfrac{x^4}{4} + y^3 x^{-4}\right] dx}$

$\Rightarrow \cancel{\dfrac{x^4}{4} + y^3 x} = c$

$\Rightarrow \int \dfrac{1}{x} + \dfrac{y^3}{x^4} dx + \int 0 dy = c$

$\Rightarrow \log x + y^3 \int x^{-4} dx = c$

$\Rightarrow \log x + y^3 \dfrac{x^{-3}}{-3} = c$

10/June/2024.

3Q. Solve $y[x^2y^2-1]dx + x[x^2y^2+1]dy = 0$

Sol: Given $y[x^2y^2-1]dx + x[x^2y^2+1]dy = 0$ —— ①

where $M = x^2y^3 - y$, $N = x^3y^2 + x$

$\therefore \frac{\partial M}{\partial y} = 3x^2y^2 - 1$, $\frac{\partial N}{\partial x} = 3x^2y^2 + 1$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ① is not exact differential equation.

$Mx - Ny = (x^2y^3 - y)x - (x^3y^2 + x)y$

$= \cancel{x^3y^3} - xy - \cancel{x^3y^3} - xy$

$= -2xy$

$I.F = \frac{1}{Mx - Ny} = \frac{1}{-2xy}$

$I.F \times ①$ $\frac{1}{-2xy}[x^2y^3 - y]dx - \frac{1}{2xy}[x^3y^2 + x]dy = 0$

$= \left[\frac{xy^2}{-2} + \frac{1}{2x}\right]dx + \left[\frac{1}{2}\frac{x^2y}{} - \frac{1}{2y}\right]dy = 0$

$\therefore$ General Solution: $\int \frac{xy^2}{-2} + \frac{1}{2x}\, dx + \int -\frac{1}{2y}\, dy = 0$

$-\frac{y^2}{2}\int x\,dx + \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\int \frac{1}{y}\,dy = c$

$-\frac{y^2}{2} \cdot \frac{x^2}{2} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$

$= -\frac{x^2y^2}{4} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$ //

Q. Solve $(2x+y)\,dx-(x+2y)\,dy=0$

Sol: Given $(2x+y)\,dx-(x+2y)\,dy=0$ —— ①

where $M=2x+y$, $N=-x-2y$

$$\frac{\partial M}{\partial y}=0+1 \quad , \quad \frac{\partial N}{\partial x}=-1$$

$$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \quad \therefore \text{① is not exact differential eq}^n$$

$$Mx+Ny=(2x+y)x+(-x-2y).y$$
$$=2x^2+\cancel{xy}-\cancel{xy}-2y^2$$
$$=2x^2-2y^2$$
$$=2(x^2-y^2)$$

$$I\cdot F=\frac{1}{Mx+Ny}=\frac{1}{2(x^2-y^2)}$$

$$eq^n\ ①\times I\cdot F=\left[(2x+y)\,dx-(x+2y)\,dy\right]\times\frac{1}{2(x^2-y^2)}=0$$

$$\Rightarrow \frac{1}{2(x^2-y^2)}(2x+y)\,dx-\frac{1}{2(x^2-y^2)}(x+2y)\,dy=0$$

$$=\frac{2x\,dx}{2(x^2-y^2)}dx+\frac{y}{2(x^2-y^2)}dx-\left(\frac{(x)}{2(x^2-y^2)}dy+\frac{2y\,dy}{2(x^2-y^2)}\right)=0$$

General Solution : $\int\limits_{y=\text{constant}} M\,dx+\int N\,[\text{Terms free from } x]\,dy=c$

$$=\frac{1}{2}\int\frac{2x}{x^2-y^2}dx+\int\frac{y}{2}\cdot\frac{1}{(x^2-y^2)}dx+\int 0\,dy=0$$

$$=\frac{1}{2}\log(x^2-y^2)+\underline{\frac{y}{2}\cdot\frac{1}{2y}\log\left(\frac{x-y}{x+y}\right)}=c$$

$= \frac{1}{2}\log(x^2-y^2) + \frac{y}{2}\cdot\frac{1}{2y}\log\left(\frac{x-y}{x+y}\right) = c$

$= \log(x^2-y^2)^{1/2} + \log\left(\frac{x-y}{x+y}\right)^{1/4} = c$

$= \log(x^2-y^2)^{1/2} + \log\left(\frac{x-y}{x+y}\right)^{1/4} = c$

$= \log(x^2-y^2)^{1/2}\cdot\log\left(\frac{x-y}{x+y}\right)^{1/4} = \log c$

$\Rightarrow (x^2-y^2)^{1/2}\cdot\left(\frac{x-y}{x+y}\right)^{1/4} = c$

Q. Solve $x^2y\,dx - (x^3+y^3)dy = 0$

Sol: Given $x^2y\,dx - (x^3+y^3)dy = 0$ — ①

where $M = x^2y$, $N = -x^3-y^3$

$\frac{\partial M}{\partial y} = x^2(1) = x^2$ $\quad \frac{\partial N}{\partial x} = -3x^2 - 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\quad \therefore$ ① is not exact differential eq^n.

$Mx + Ny = (x^2y)x + (-x^3-y^3)y$

$= \cancel{x^3y} - \cancel{x^3y} - y^4$

$= -y^4$

$\therefore I.F = \frac{1}{Mx+Ny} = \frac{1}{-y^4}$

I.F x equation ① $\Rightarrow \frac{1}{-y^4}(x^2y)dx - \frac{1}{-y^4}(x^3+y^3)dy = 0$

$= -\frac{x^2}{y^3}dx + \left[\frac{x^3}{y^4} + \frac{1}{y}\right]dy = 0$

General Solution : $\int_{y=const} M\,dx + \int N\,[\text{terms free from } x]\,dy = 0$

$$\int -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = 0$$

$$-\frac{1}{y^3}\int x^2\,dx + \log y = c$$

$$-\frac{1}{y^3}\cdot\frac{x^3}{3} + \log y = c$$

$$\Rightarrow -\frac{x^3}{3y^3} + \log y = c$$

$$\Rightarrow \log y - \frac{x^3}{3y^3} = c$$

H/W

6Q. $xy\,dx - (x^2+2y^2)\,dy = 0$ · Solve this differential eq^n.

Sol. Given $xy\,dx - (x^2+2y^2)\,dy = 0$ —— ①

where $M = xy$, $N = -x^2-2y^2$

$$\frac{\partial M}{\partial y} = x(1) = x \quad , \quad \frac{\partial N}{\partial x} = -2x - 0 = -2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$ ∴ ① is not exact differential eq^n

$$Mx+Ny = (xy)\,x + (-x^2-2y^2)\,y$$

$$= \cancel{x^2y} - \cancel{x^2y} - 2y^3$$

$$= -2y^3$$

$$\therefore \ I.F = \frac{1}{Mx+Ny} = \frac{1}{-2y^3}$$

$I.F \times eq^n (1) \Rightarrow -\frac{1}{2y^3}[xy\,dx] - \frac{1}{2y^3}[-x^2-2y^2]$

$\Rightarrow -\frac{x}{2y^2}\,dx + \frac{x^2}{2y^3}\,dy + \frac{1}{y}\,dy$

$= c.$

$\therefore$ General Solution : $\int\limits_{y=const} M\,dx + \int\limits_{\text{terms free from } x} N\,dy$

$\int -\frac{x}{2y^2}\,dx + \int \frac{1}{y}\,dy = c$

$\frac{1}{2y^2}\int -x\,dx + \log y = c$

$\frac{-1}{2y^2}\int x\,dx + \log y = c$

$\frac{-1}{2y^2} \times \frac{x^2}{2} + \log y = c$

$-\frac{x^2}{4y^2} + \log y = c$

$\therefore \log y - \frac{x^2}{4y^2} = c$, is the solution //.

→ Type-III : If given non exact differential equation $y f(x,y)dx + x g(x,y)dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an integral factor.

Q1 Solve $y[x^2y^2-1]dx + x[x^2y^2+1]dy = 0$ —— ①

Sol: Given $M = x^2y^3 - y$, $N = x^3y^2 + x$ ——

$$\frac{\partial M}{\partial y} = 3x^2y^2 - 1 \quad , \quad \frac{\partial N}{\partial x} = 3x^2y^2 + 1$$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\quad \therefore$ ① is not exact differential eq^n.

$$Mx - Ny = (x^2y^3 - y)x - (x^3y^2 + x)y$$
$$= \cancel{x^3y^3} - xy - \cancel{x^3y^3} - xy$$
$$= -2xy$$

$$I.F = \frac{1}{Mx-Ny} = \frac{1}{-2xy}$$

① × I.F $\Rightarrow (x^2y^3 - y)\frac{1}{-2xy}dx + (x^3y^2+x)\frac{1}{-2xy}dy = 0$

$$= \left[-\frac{xy^2}{2} + \frac{1}{2x}\right]dx - \left[\frac{x^2y}{2} - \frac{1}{2y}\right]dy = 0$$

General Solution : $\int\limits_{y=\text{const}} M dx + \int N[\text{Terms free from } x]dy = c$.

$$\int -\frac{xy^2}{2}dx + \frac{1}{2x}dx + \int \frac{1}{2y}dy = c$$

$$-\frac{y^2}{2}\int x\,dx + \frac{1}{2}\int\frac{1}{x}dx + \frac{1}{2}\int\frac{1}{y}dy = c$$

$$-\frac{y^2}{2}\cdot\frac{x^2}{2} + \frac{1}{2}\log x + \frac{1}{2}\log y = c$$

$$\Rightarrow -\frac{x^2y^2}{2} + \log x + \log y = 2c$$

$$\Rightarrow -\frac{x^2y^2}{2} + \log xy = 2c$$

$$\Rightarrow \log xy - \frac{x^2y^2}{2} = c \quad //$$

Q2. Solve $y[1+xy]dx + x[1-xy]dy = 0$.

Sol: Given $y[1+xy]dx + x[1-xy]dy = 0$ — ①

where $M = y + xy^2$, $N = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$, $\frac{\partial N}{\partial x} = 1 - 2xy$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ ① is not exact differential equation.

$$Mx - Ny = (y + xy^2)x - (x - x^2y)y$$

$$= \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$$

$$= 2x^2y^2$$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}$$

$$① \times I.F = \frac{1}{2x^2y^2}(y + xy^2)\,dx + \frac{1}{2x^2y^2}(x - x^2y)\,dy = 0$$

$$\Rightarrow \frac{1}{2x^2y}dx + \frac{1}{2x}dx + \frac{1}{2xy^2}dy - \frac{1}{2y}dy = 0$$

$\therefore$ General Solution : $\int M dx + \int N$ [terms free from x] $= c$
$y = const$

$$\int \frac{1}{2x^2y} dx + \frac{1}{2x} dx + \int -\frac{1}{2y} dy = c$$

$$\Rightarrow \frac{1}{2y} \int \frac{1}{x^2} dx + \frac{1}{2}\int \frac{1}{x} dx - \frac{1}{2} \int \frac{1}{y} dy = c$$

$$\Rightarrow \frac{1}{2y} \int x^{-2} dx + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$x^n = \frac{x^{n+1}}{n+1}$$

$$\Rightarrow \frac{1}{2y} \times \frac{-1}{x} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\int x^{-2} = \frac{x^{-2+1}}{-2+1}$$

$$\Rightarrow \frac{-1}{2xy} + \frac{1}{2}\log\left(\frac{x}{y}\right) = 2C$$

$$\int x^{-2} = \frac{x^{-1}}{-1} = \frac{-1}{x}$$

$$\Rightarrow -\frac{1}{xy} + \log \frac{x}{y} = \tilde{C}$$

$$\Rightarrow \log\frac{x}{y} - \frac{1}{xy} = \tilde{C}$$

$[1+2xy - x^3y^3]\, x\,dy = 0$

Q3. Solve $[2xy+1]y\, dx + [1+2xy - x^3y^3]\, dy = 0$ —— (1)

Sol. Given $y[2xy+1]\, dx + x[1+2xy - x^3y^3]$

Where $M = 2xy^2 + y$, $N = x + 2x^2y - x^4y^3$

$$\frac{\partial M}{\partial y} = 2x(2y)+1 = 4xy+1$$

$$\frac{\partial N}{\partial x} = 1 + 2(2x)y - 4x^3y^3 = 1 + 4xy - 4x^3y^3$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not exact differential equation.

$$M_x - N_y = [2xy^2 + y]x - [x + 2x^2y - x^4y^3]y$$
$$= 2x^2y^2 + xy - xy - 2x^2y^2 + x^4y^4$$
$$= x^4y^4$$

$$\therefore I.F = \frac{1}{Mx - Ny} = \frac{1}{x^4y^4}$$

$$① \times I.F \Rightarrow \frac{1}{x^4y^4}[2xy^2 + y]dx + \frac{1}{x^4y^4}[x + 2x^2y - x^4y^3]\frac{dy}{=0}$$

$$\Rightarrow \left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right)dx + \left(\frac{1}{x^3y^4} + \frac{2}{x^2y^3} - \frac{1}{y}\right)\frac{dy}{=0}$$

$\Rightarrow \therefore$ General Solution : $\int\limits_{y=const} M\,dx + \int N$ [Term free from x] $= c$

$$\Rightarrow \int \left(\frac{2}{x^3y^2} + \frac{1}{x^4y^3}\right)dx + \int -\frac{1}{y}\,dy = c$$

$$\Rightarrow \frac{2}{y^2}\int \frac{1}{x^3} + \frac{1}{y^3}\int \frac{1}{x^4}\,dx + \int -1/y\,dy = c$$

$$\Rightarrow \frac{2}{y^2}\int \frac{1}{x^3} + \frac{1}{y^3}\int \frac{1}{x^4}\,dx + -1\int \frac{1}{y}\,dy = c$$

$$\Rightarrow \frac{\cancel{2}}{y^2} \times \frac{1}{-\cancel{2}x^2} + \frac{1}{y^3} \times \frac{1}{-3x^3} - \log y = c$$

$$\Rightarrow -\frac{1}{x^2y^2} - \frac{1}{3x^3y^3} - \log y = c$$

//

$$y^{-1} = \frac{y^{-1+1}}{-1+1}$$

$$x^{-3} = \frac{x^{-3+1}}{-3+1} = \frac{x^{-2}}{-2}$$

$$x^{-4} = \frac{x^{-4+1}}{-4+1} = \frac{x^{-3}}{-3}$$

H/w.

Q4. Solve $y[xy+2x^2y^2]dx + x[xy - x^2y^2]dy = 0$.

Sol: Given. $y[xy+2x^2y^2]dx + x[xy-x^2y^2]dy = 0$ ①

$[xy^2+2x^2y^3]dx + [x^2y - x^3y^2]dy = 0$.

$M = xy^2 + 2x^2y^3 \qquad N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = x(2y) + 2x^2(3y^2) \qquad \frac{\partial N}{\partial x} = 2xy - (3x^2)y^2$

$= 2xy + 6x^2y^2 \qquad = 2xy - 3x^2y^2$

$\therefore \boxed{\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}}$.

$Mx - Ny = [xy^2 + 2x^2y^3]x - [x^2y - x^3y^2]y$

$= \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} - x^3y^3$

$= x^3y^3$

$\therefore I.F = \frac{1}{Mx-Ny} = \frac{1}{x^3y^3}$

① $\times$ I.F $\Rightarrow \frac{1}{x^3y^3}[xy^2 + 2x^2y^3]dx + \frac{1}{x^3y^3}[x^2y - x^3y^2]dy = 0$

$\Rightarrow \left[\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left[\frac{1}{xy^2} - \frac{1}{y}\right]dy = 0$.

$\therefore$ General Solution : $\int\limits_{y=\text{const}} M dx + \int N$ (terms free from x) $= C$

$\int\left[\frac{1}{x^2y} + \frac{2}{x}\right]dx + \int\left[\cancel{\frac{1}{xy^2}} - \frac{1}{y}\right]dy = c$

$\frac{1}{y}\int\frac{1}{x^2}dx + 2\int\frac{1}{x}dx - \int\frac{1}{y}dy = c$

$\Rightarrow \frac{1}{y}\left[-\frac{1}{x}\right] + 2\log x - \log y = c$.

$\Rightarrow -\frac{1}{xy} + 2\log x - \log y = c$.

$\therefore$ General Solution is $-\frac{1}{xy} + 2\log x - \log y = c$

Type-V : • If it is a non exact differential equation .
• If it is a non homogeneous differential equation -
• If it is not in the form of $yf(x,y)\,dx + xg(x,y)\,dy = 0$
• and M is small $\Rightarrow \dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = f(y)$ or constant

$$\Rightarrow I \cdot F = e^{\int f(y)\,dy}$$.

Type IV : • If it is a non exact differential equation -
• It is a non-homogeneous differential equation .
• If it is not in the form of $yf(x,y)\,dx + xg(x,y)\,dy = 0$
• and N is small $\Rightarrow \dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ or constant .

$$\Rightarrow I \cdot f = e^{\int f(x)\,dx}$$.

→ Type IV and V :

→ If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = f(x) \Rightarrow I \cdot F = e^{\int f(x)dx}$

(N is small)

→ If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial m}{\partial y}}{m} = f(y) \Rightarrow I \cdot F = e^{\int f(y)dy}$

(M is small)

Q1:- Solve $2xy\,dy - (x^2+y^2+1)dx = 0$

Sol: Given $2xy\,dy - (x^2+y^2+1)dx = 0$ —— ①

$m = -x^2 - y^2 - 1, \quad N = 2xy$

$\dfrac{\partial m}{\partial y} = -2y, \quad \dfrac{\partial N}{\partial x} = 2y$

$\therefore \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ $\quad \therefore$ ① is not exact differential eqn

→ $\dfrac{\dfrac{\partial m}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \dfrac{-2y-2y}{2xy} = \dfrac{-4y}{2xy} = -\dfrac{2}{x} = f(x)$

$I \cdot F = e^{\int f(x)dx}$

$\therefore I \cdot F = e^{\int -\frac{2}{x}dx}$

$= e^{-2\int \frac{1}{x}dx}$

$= e^{-2\log x}$

$= e^{\log x^{-2}}$

$I \cdot F = x^{-2} = \dfrac{1}{x^2}$

$I \cdot F \times eq^n ① \Rightarrow \dfrac{1}{x^2}\left[2xy\,dy - (x^2+y^2+1)dx\right] = 0$

$\Rightarrow \frac{1}{x^2}[-x^2-y^2-1]\,dx + \frac{1}{x^2}[2xy]\,dy = 0$

$\Rightarrow \left[-1-\frac{y^2}{x^2}-\frac{1}{x^2}\right]dx + \frac{2y}{x}\,dy = 0$

General Solution $\Rightarrow$

$\int M\,dx + \int N\,dy = c$

$\int -1-\frac{y^2}{x^2} - x^{-2}]\,dx + \int 0\,dy = c$

$\Rightarrow -x - \frac{y^2x^{-1}}{-1} - \frac{x^{-1}}{-1} = c$

$\Rightarrow -x - \frac{y^2}{x} + \frac{1}{x} = c$

$\Rightarrow \frac{1}{x} - x - \frac{y^2}{x} = c$ //

$[xy^3+y]\,dx + 2[x^2y^2+x+y^4]\,dy = 0$

Q2. Solve $[xy^3+y]\,dx + 2[x^2y^2+x+y^4]\,dy = 0$ — ①

Sol. Given $[xy^3+y]\,dx + 2[x^2y^2+x+y^4]\,dy$

$M = xy^3+y$, $N = 2x^2y^2+2x+2y^4$

$\frac{\partial M}{\partial y} = 3xy^2+1$, $\frac{\partial N}{\partial x} = 2(2x)y^2 + 2(1) + 0$

$= 4xy^2+2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ ① is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{3xy^2+1-4xy^2-2}{2x^2y^2+2x+2y^4} = \frac{-xy^2-1}{2x^2y^2+2x+2y^4}$$

$\neq f(x)$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{4xy^2+2-3xy^2+1}{xy^3+y} = \frac{xy^2+1}{y[xy^2+1]} = \frac{1}{y} = f(y)$$

$$\therefore I.F = e^{\int f(y)dy}$$

$$= e^{\int \frac{1}{y}dy}$$

$$= e^{\log y}$$

$$= y$$

$$\because e^{\log x} = x, \quad e^{\log y} = y$$

$$(1) \times IF = y[xy^3+y]dx + 2y[x^2y^2+x+y^4]dy = 0$$

$$[xy^4+y^2]dx + [2x^2y^3+2xy+2y^5]dy = 0$$

General solution : $\int (xy^4+y^2)\,dx + \int 2y^5\,dy = c$

$$\Rightarrow y^4\int x\,dx + y^2\int dx + 2\int y^5 dy = c$$

$$\Rightarrow y^4\frac{x^2}{2} + y^2x + \cancel{2}\frac{y^6}{\cancel{6}\,3} = c$$

$$\Rightarrow \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$$

$$\Rightarrow \frac{3x^2y^4+6xy^2+2y^6}{6} = c$$

$$\Rightarrow 3x^2y^4+6xy^2+2y^6 = c$$

Q3. Solve $(3xy-2ay^2)dx+(x^2-2axy)dy=0$

Sol. Given $(3xy-2ay^2)dx+(x^2-2axy)dy=0$ —— ①

$M=3xy-2ay^2$, $N=x^2-2axy$

$\frac{\partial M}{\partial y}=3x(1)-2a(2y)$ $\quad$ $\frac{\partial N}{\partial x}=2x-2a(1)y$

$=3x-4ay$ $\quad$ $=2x-2ay$

$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ $\therefore$ ① is not exact differential eqn.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{3x-4ay-2x+2ay}{x^2-2axy}=\frac{x-2ay}{x(x-2ay)}=\frac{1}{x}=f(x)$$

$$\therefore I.F=e^{\int f(x)dx}=e^{\int\frac{1}{x}dx}=e^{\log x}=x$$

$\therefore$ ① $\times$ I.F $= x(3xy-2ay^2)dx+x(x^2-2axy)dy=0$

$(3x^2y-2axy^2)dx+(x^3-2ax^2y)dy=0$

General solution: $\int(3x^2y-2axy^2)dx+\int 0\,dy=0$

$\Rightarrow 3y\int x^2-2ay^2\int x\,dx=C$

$\Rightarrow \cancel{3}y\frac{x^3}{\cancel{3}}-\cancel{2}ay^2\cdot\frac{x^2}{\cancel{2}}=C$

$\Rightarrow x^3y-ax^2y^2=C$

//

14/06/2024.

Q4: Solve $(x^2+y^2+2x)dx+2ydy=0$.

Sol: Given $(x^2+y^2+2x)dx+2ydy=0$ — ①.

where $M=x^2+y^2+2x$, $N=2y$.

$$\frac{\partial M}{\partial y}=0+2y+0 \quad \frac{\partial N}{\partial x}=0$$

$$\therefore \frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x} \qquad \therefore \text{① is not exact differential eq}^n.$$

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{2y-0}{2y}=\frac{2y}{2y}=1$$

$$I.F=e^{\int f(x)dx}=e^{\int 1dx}=e^x$$

$$① \times I.F \Rightarrow e^x[x^2+y^2+2x]dx+e^x.2ydy=0$$

$$\Rightarrow [e^x x^2+e^x y^2+e^x 2x]dx+e^x 2ydy=0$$

General Solution $\int_{y=\text{const}} M\,dx+\int N\,[\text{Terms free from } x]\,dy=c$

$$\int(e^x x^2+e^x y^2+e^x 2x)dx+\int 0\,dy=c$$

$$\int x^2 e^x dx+y^2\int e^x dx+2\int xe^x dx=c$$

$$\cancel{\Rightarrow x^2e^x-2e^x+2e^x+y^2e^x+2(xe^x-e^x)=c}$$

$$\cancel{\Rightarrow xe^x-e^x+y^2e^x+2xe^x-2e^x=c}$$

$$\cancel{\Rightarrow 3xe^x-3e^x+y^2e^x=c}$$

$$\Rightarrow x^2e^x-\cancel{2}xe^x+2e^x+y^2e^x+2\cancel{x}e^x-2e^x=c$$

$$\Rightarrow x^2e^x+y^2e^x=c$$

$\therefore$ The required solution is $x^2e^x+y^2e^x=c$ //.

Q5. $(x^3-2y^2)dx + 2xy\,dy = 0$.

Sol: Given $(x^3-2y^2)dx + 2xy\,dy = 0$ ——— ①

where $M = x^3-2y^2$, $N = 2xy$

$\frac{\partial M}{\partial y} = 0-2(2y) = -4y$ $\qquad$ $\frac{\partial N}{\partial x} = 2(1)y = 2y$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\qquad$ $\therefore$ ① is not exact differential eqn.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-4y-2y}{2xy} = \frac{-\cancel{6}^{3}\cancel{y}}{\cancel{2}x\cancel{y}} = \frac{-3}{x}$$

$$I.F = e^{\int f(x)dx} = e^{\int -\frac{3}{x}dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$$

$$① \times I.F = \frac{1}{x^3}(x^3-2y^2)dx + \frac{1}{x^3}(2xy)dy = 0$$

$$= \left(1-\frac{2y^2}{x^3}\right)dx + \left(\frac{2y}{x^2}\right)dy = 0$$

$\therefore$ General Solution: $\int\limits_{y=const} M\,dx + \int N \text{ (terms free from } x) dy = c$

$$\int\limits_{y=const}\left(1-\frac{2y^3}{x^3}\right)dx + \int 0\,dy = c$$

$$\int 1\,dx - 2y^2\int\frac{1}{x^3}dx = c$$

$$x - 2y^2\left[\frac{x^{-3+1}}{-3+1}\right] = c$$

$$x - \cancel{2}y^2\left[\frac{x^{-2}}{-\cancel{2}}\right] = c \Rightarrow x + y^2 x^{-2}$$

$$x + \frac{y^2}{x^2} = c$$

Q6. Solve $(y^4+2y)dx+[xy^3+2y^4-4x]dy=0$.

Sol: Given $(y^4+2y)dx+[xy^3+2y^4-4x]dy=0$ — (1)

where $M = y^4+2y$, $N = xy^3+2y^4-4x$.

$$\frac{\partial M}{\partial y} = 4y^3+2(1) = 4y^3+2 \qquad \frac{\partial N}{\partial x} = 1(y^3)+0-4(1) = y^3-4$$

∴ $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ∴ (1) is not exact differential eqn.

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = \frac{4y^3+2-y^3+4}{xy^3+2y^4-4x} = \frac{3y^3+6}{xy^3+2y^4-4x} \neq f(x).$$

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{xy^3+2y^4-4x-y^4-2y}{y^4+2y} = \frac{xy^3+2y^4-4x-y^4-2y}{y^4+2y}$$

$$= \frac{xy^3+y^4-2y-4x}{y^4+2y}$$

$$= xy^3-4x-(y^4$$ ×

$$\Rightarrow \frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = \frac{y^3-4-4y^3-2}{y^4+2y} = \frac{-3y^3-6}{y^4+2y}$$

$$\frac{-3\cancel{(y^3+2)}}{y\cancel{(y^3+2)}} = \frac{-3}{y} = f(y)$$

$$I.F = e^{\int f(y)dy}$$

$$= e^{\int -\frac{3}{y}dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

① $\times$ I.F $= \frac{1}{y^3}[y^4 - 2y]\,dx + \frac{1}{y^3}[xy^3 + 2y^4 - 4x]\,dy = 0$

$\Rightarrow \left[y + \frac{2}{y^2}\right] dx + \left[x + 2y - \frac{4x}{y^3}\right] dy = 0$

General Solution $\int\limits_{y=const} M\,dx + \int N$ (terms free from x) $dy = c$.

$\int \left(y + \frac{2}{y^2}\right) dx + \int a + 2y\,dy = c$

$y\int dx + \frac{2}{y^2}\int dx + 2\int y\,dy = c$

$yx + \frac{2}{y^2}x + \cancel{2}\cdot\frac{y^2}{\cancel{2}} = c$

$xy + \frac{2x}{y^2} + y^2 = c \Rightarrow \cancel{xy^3 + 2x + y^4 = c}$ //.

//.

Q7. Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$.

Sol: Given $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$

where $M = 3x^2y^4 + 2xy$, $N = 2x^3y^3 - x^2$

$\frac{\partial M}{\partial y} = 3x^2(4y^3) + 2x(1)$, $\frac{\partial N}{\partial x} = 2(3x^2)y^3 - 2x$

$= 12x^2y^3 + 2x$ $\quad = 6x^2y^3 - 2x$.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{3x^2y^4 + 2xy}$$

$$= \frac{-6x^2y^3 - 4x}{3x^2y^4 + 2xy} = \frac{-2\cancel{x}(3\cancel{x}^2y^3 + 2\cancel{x})}{y(3x^2y^3 + 2x)}$$

$$= \frac{-2}{y}.$$

$I.F = e^{\int f(y)dy} = e^{-2\int \frac{1}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$

$① \times I.F = \frac{1}{y^2}[3x^2y^4 + 2xy]dx + \frac{1}{y^2}[2x^3y^3 - x^2]dy = 0$

$= \left[3x^2y^2 + \frac{2x}{y}\right]dx + \left[2x^3y - \frac{x^2}{y^2}\right]dy = 0$

General Solution : $\int\limits_{y=const} M dx + \int N [\text{terms free from } x]\, dy = c$

$\Rightarrow \int(3x^2y^2 + \frac{2x}{y})dx + \int 0\, dy = c$

$\Rightarrow 3y^2\int x^2 dx + \frac{2}{y}\int x\, dx = c$

$\Rightarrow \cancel{3}y^2 \cdot \frac{x^3}{\cancel{3}} + \frac{\cancel{2}}{y} \cdot \frac{x^2}{\cancel{2}} = c$

$\Rightarrow x^3y^2 + \frac{x^2}{y} = c$ //.

Q8. Solve $(y+y^2)dx + xy\,dy = 0$.

Sol: Given $(y+y^2)dx + xy\,dy = 0$ —①

where $M = y+y^2$ $N = xy$.

$\frac{\partial M}{\partial y} = 1+2y$ $\frac{\partial N}{\partial x} = y$.

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\therefore$ ① is not exact differential eq^n.

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{y-1-2y}{y+y^2} = \frac{-y-1}{y+y^2} = \frac{-(\cancel{y+1})}{y(\cancel{1+y})} = \frac{-1}{y}$

$= f(y)$

$I.F = e^{\int f(y)dy} = e^{-\int \frac{1}{y}dy} = e^{-\log y} = e^{\log y^{-1}} = \frac{1}{y}$..

① $\times$ I.F $= \frac{1}{y}(y+y^2)\,dx + \frac{1}{y}(xy\,dy) = 0$

$(1+\frac{1}{y})\,dx + x\,dy = 0$

General Solution $= \int M\,dx + \int N[\text{Terms free from } x]\,dy = 0$

$= \int(1+\frac{1}{y})\,dx + \int 0\,dy = 0$

$\Rightarrow \int 1\,dx + \frac{1}{y}\int dx \quad \Rightarrow x + \frac{x}{y} = c.$

$\Rightarrow \cancel{x + \frac{1}{y}xy + c}$

$\Rightarrow \cancel{x + \frac{xy}{y} + c}$ //.

Q9. Solve $y[2xy + e^x]\,dx - e^x\,dy = 0$.

Sol: Given $[2xy^2 + e^x y]\,dx - e^x\,dy = 0$ — ①

$M = 2xy^2 + e^x y \qquad N = -e^x$.

$\frac{\partial M}{\partial y} = 2x(2y) + e^x(1) \qquad \frac{\partial N}{\partial x} = -e^x$

$= 4xy + e^x$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \therefore$ ① is not exact.

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \frac{-e^x - 4xy - e^x}{2xy^2 + e^x y} = \frac{-2e^x - 4xy}{2xy^2 + e^x y}$$

$$= \frac{-2(e^x + \cancel{2xy})}{y(\cancel{2xy} + e^x)} = \frac{-2}{y}.$$

$\therefore$ I.F $= e^{\int f(y)dy} = e^{\int \frac{-2}{y}} = e^{-2\int \frac{1}{y}} = e^{-2\log y} = e^{\log y^{-2}} = y^{-2} = \frac{1}{y^2}$

~~Genera~~

(1) $\times$ I.F $= \frac{1}{y^2}(2xy^2 + e^x y)dx - \frac{1}{y^2} e^x dy = 0$

$$\left(2x + \frac{e^x}{y}\right)dx - \frac{e^x}{y^2} dy = 0$$

General Solution $= \int M dx + \int N$ (terms free from x) $dy = c$

$$\int 2x + \frac{e^x}{y} dx + \int 0\, dy = c$$

$$\int 2x dx + \frac{1}{y}\int e^x dx = c$$

$$\cancel{2}\frac{x^2}{\cancel{2}} + \frac{1}{y} e^x = c$$

$$x^2 + \frac{e^x}{y} = c \quad //$$

② Linear Differential Equation :

An equation of the form $\boxed{\frac{dy}{dx} + Py = Q(x)}$ is called as Linear differential equation.

$\Rightarrow$ $I.F = e^{\int P(x)dx}$ $\quad e^{\int Pdx}$

$\Rightarrow$ General Solution $\Rightarrow$ $y(I.F) = \int Q(I.F)dx + c$

$\Rightarrow$ Similarly $\Rightarrow$ $\boxed{\frac{dx}{dy} + Px = Q(y)}$

$\Rightarrow$ $I.F = e^{\int P(y)dy}$ $\quad e^{\int Pdy}$

$\Rightarrow$ General Solution $\Rightarrow$ $x(I.F) = \int Q(I.F)dy + c$.

Q1. $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$

Sol Given $\frac{dy}{dx} + Py = Q(x)$

$P = \frac{y}{x\log x}$, $Q(x) = \frac{\sin 2x}{\log x}$

$I.F = e^{\int Pdx} = e^{\int \frac{1}{x\log x}dx}$

let $\log x = t$

$\frac{1}{x}dx = dt$

$= e^{\int \frac{1}{t}dt} = e^{\log t} = t = \log x$.

$\because \int \sin 2x = \frac{-\cos 2x}{2}$

$\therefore$ General Solution; $y(I.F) = \int Q(I.F)dx + c$

$y\log x = \int \frac{\sin 2x}{\cancel{\log x}} \cancel{\log x}\, dx + c$

(or) $y\log x = \int 2\sin x\cos x\, dx + c$

$y\log x = \frac{(\sin^2 x)}{2} + c$.

$y\log x = \frac{-\cos 2x}{2} + c$

$y\log x + \frac{\cos 2x}{2} + c$ //.

21/06/2024.

Q.2 Solve $(1+x^2)y'+y+1=0$.

Sol: Given $(1+x^2)y'+y+1=0$

$\Rightarrow (1+x^2)y'+y=-1$

$$y'+\frac{y}{(1+x^2)}=\frac{-1}{1+x^2}$$

$$\frac{dy}{dx}+\frac{1}{1+x^2}y=\frac{-1}{1+x^2}$$

$\Rightarrow \left[\frac{dy}{dx}+Py=Q(x) \rightarrow \text{L.D equation in } x\right]$

$$P=\frac{1}{1+x^2} \quad Q=\frac{-1}{1+x^2}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}.$$

$\therefore$ General Solution; $y(I.F)=\int Q(I.F)dx+c$

$$y\cdot e^{\tan^{-1}x}=\int \frac{-1}{1+x^2}e^{\tan^{-1}x}dx+c$$

$$y\cdot e^{\tan^{-1}x}=-\int \frac{1}{1+x^2}e^{\tan^{-1}x}dx+c$$

Put $\tan^{-1}x=t$

$$\frac{1}{1+x^2}dx=dt$$

$$y\cdot e^{\tan^{-1}x}=\int -e^{t}dt$$

$$ye^{\tan^{-1}x}=-e^{\tan^{-1}x}+c.$$

3Q. Solve $xy'+y+4=0$

Sol: Given $xy'+y+4=0$

$$x\frac{dy}{dx}+y+4=0$$

Dividing throughout by x;

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{-4}{x}$ $\quad [\frac{dy}{dx} + Py = Q$; linear Differential eqⁿ in x].

$P = \frac{1}{x}$, $Q = \frac{-4}{x}$

Integrating factor $= e^{\int P dx}$

$= e^{\int \frac{1}{x}}$

$= e^{\log x} = x$.

$\therefore$ General Solution : $y(I \cdot F) = \int Q (I \cdot F) dx + C$

$y \cdot x = \int -\frac{4}{\cancel{x}} \cancel{x} \, dx + C$

$xy = -4 \int dx + C$

$xy = -4x + C$

$xy + 4x = C$

$x(y+4) = C$ //.

Q4. Solve $(x+2y^3) \frac{dy}{dx} = y$.

Sol: Given $(x+2y^3) \frac{dy}{dx} = y$

$(x+2y^3) = \frac{dx}{dy} y$

$\frac{(x+2y^3)}{y} = \frac{dx}{dy}$

$\frac{dx}{dy} = \frac{x}{y} + \frac{2y^3}{y}$

$\frac{dx}{dy} = \frac{x}{y} + 2y^2$

$\boxed{\frac{dx}{dy} - \frac{1}{y}(x) = 2y^2}$ $\quad [\frac{dx}{dy} + P(x) = Q(y)$; linear Differential eqⁿ in y] where $P = -\frac{1}{y}$, $Q = 2y^2$

Integrating factor $= e^{\int P dy}$

$= e^{\int -\frac{1}{y} dy}$

$= e^{-\log y}$

$= e^{\log y^{-1}}$

$= y^{-1}$

$= \frac{1}{y}$.

$\therefore$ General Solution: $x(I.F) = \int Q(I.F)\, dy + c$.

$$x\left(\frac{1}{y}\right) = \int 2y^{\cancel{2}} \cdot \frac{1}{\cancel{y}}\, dy + c$$

$$\frac{x}{y} = \int 2y\, dy + c$$

$$\frac{x}{y} = \cancel{2} \times \frac{y^2}{\cancel{2}} + c$$

$$\frac{x}{y} - y^2 = c \;//$$

5Q. Solve $\frac{dx}{dy} + \frac{2y}{1+y^2} x = \frac{1}{(1+y^2)^2}$

Sol: Given $\frac{dx}{dy} + \frac{2y}{1+y^2} x = \frac{1}{(1+y^2)^2}$

[$\frac{dx}{dy} + P(x) = Q(y)$; Linear differential eq^n in y]

$P = \frac{2y}{1+y^2}$; $Q = \frac{1}{(1+y^2)^2}$

Integrating factor $= e^{\int P dy} = e^{\int \frac{2y}{1+y^2} dy} = e^{\log(1+y^2)} = 1+y^2$

$\therefore$ General Solution: $x(I.F) = \int Q(I.F)\, dy + c$

$$x(1+y^2) = \int \frac{1}{(1+y^2)^{\cancel{2}}} \cdot \cancel{(1+y^2)} + C$$

$$x(1+y^2) = \int \frac{1}{1+y^2} + c$$

$\Rightarrow x(1+y^2) = \tan^{-1}y + c$.

6Q. Solve $x\frac{dy}{dx} + y = \log x$.

Sol: Given $x\frac{dy}{dx} + y = \log x$

$$\frac{x}{x}\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}.$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}\log x \quad \left[\frac{dy}{dx} + Py = Qx \text{ ; linear differential eq}^n \text{ in } x\right]$$

$P = \frac{1}{x}$, $Q = \frac{\log x}{x}$

Integrating factor: $e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$.

∴ General Solution: $y(I.F) = \int Q(I.F)dx + c$

$$y(x) = \int \frac{\log x}{\cancel{x}} \cdot \cancel{x}\, dx + c$$

$$y(x) = \int \log x \cdot dx + c$$

$$y(x) = \int 1 \cdot \log x\, dx + c \qquad 1 \cdot \log x.$$

$$y(x) = \cancel{\log x \cdot x - \frac{1}{x}\frac{x^2}{2} + c}$$

$$y(x) = \cancel{x\left[\log x - \frac{x}{2}\right]} + c.$$

$$y(x) = x\log x - x + c$$

$$y(x) = x(\log x - 1) + c.$$

$$\boxed{\int \log x = x\log x - x + c}$$

(3) Bernouillis Differential Equations.

[Equations reduced to linear Differential equation]

An equation of the type $\boxed{\frac{dy}{dx} + Py = Q \cdot y^n}$ is called Bernouillis differential equation in y.

Similarly $\boxed{\frac{dx}{dy} + Px = Q \cdot x^n}$ is called Bernouillis differential equation in x.

Q1. Solve $x\frac{dy}{dx} + y = x^3 y^6$.

Sol: Given $x\frac{dy}{dx} + y = x^3 y^6$

Divide throughout by x.

$\Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2 y^6$

Divide throughout by y^6.

$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} + \frac{1}{y^6}\frac{y}{x} = \frac{1}{y^6}(x^2 y^6)$

$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2$ [always select this as z].

Let $y^{-5} = z$

$-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$

$y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dz}{dx}$.

$\Rightarrow -\frac{1}{5}\frac{dz}{dx} + \frac{1}{x}z = x^2$

This is of the form $\boxed{\frac{dz}{dx} - \frac{5z}{x} = -5x^2}$

$\Rightarrow \frac{dz}{dx} + Pz = Qy \Rightarrow$ L.D eq^n in x.

where $P = -\frac{5}{x}$, $Q = -5x^2$

$I.F = e^{\int P dx} = e^{\int -\frac{5}{x} dx} = e^{-5\log x} = e^{\log x^{-5}} = x^{-5} = \frac{1}{x^5}$.

$\therefore$ General Solution: $z(I.F) = \int Q(I.F)\,dx + C$

$\Rightarrow z \cdot \frac{1}{x^5} = \int -5x^2 \cdot \frac{1}{x^5}\,dx + C$

$\Rightarrow y^{-5} \cdot \frac{1}{x^5} = -5 \int x^{-3} dx + C$

$\Rightarrow y^{-5} \frac{1}{x^5} = -5 \frac{x^{-2}}{-2} + C$

$\Rightarrow y^{-5} \frac{1}{x^5} = \frac{5x^{-2}}{2} + C$

$\Rightarrow \frac{1}{x^5 y^5} = \frac{5}{2x^2} + C$ //

Q2. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$.

Sol: Given $\frac{dy}{dx} + \frac{y}{x} = y^2 x \sin x$

Divide throughout by y^2.

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y^2}\frac{y}{x} = \frac{y^2 x \sin x}{y^2}$

$\Rightarrow y^{-2}\frac{dy}{dx} + \frac{y^{-1}}{x} = x \sin x$ always select y^{-1} as z.

$\Rightarrow y^{-2}\frac{dy}{dx} + \frac{1}{xy} = x \sin x$

$\Rightarrow y^{-2}\frac{dy}{dx} + y^{-1}\frac{1}{x} = x \sin x$

let $y^{-1} = z$

$-1 y^{-2}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow y^{-2}\frac{dy}{dx} = -1\frac{dz}{dx}$. $\left[\frac{dz}{dx} + Pz = Qx\right.$ linear Differential eq^n in $x\left.\right]$.

$\Rightarrow -1\frac{dz}{dx}+\frac{1}{x}z = x\sin x$

$$\boxed{\frac{dz}{dx}-\frac{1}{x}z = -x\sin x}$$

where $P = -\frac{1}{x}$, $Q = -x\sin x$.

Integrating factor: $e^{\int P dx} = e^{\int \frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$.

$\therefore$ General solution: $z(I\cdot F) = \int Q(I\cdot F)dx + c$

$$z\left(\frac{1}{x}\right) = \int -\cancel{x}\sin x \cdot \frac{1}{\cancel{x}}dx + c$$

$$\frac{z}{x} = -\int \sin x\, dx + c$$

$$\frac{z}{x} = -[-\cos x] + c$$

$$\frac{z}{x} = \cos x \mp c.$$

$$\frac{z}{x} - \cos x \mp c.$$

$$\frac{y^{-1}}{x} - \cos x \mp c$$

$$\frac{1}{xy} - \cos x \mp c \quad //.$$

Q3. Solve $\frac{dy}{dx}[x^2y^3+xy] = 1$.

Sol: Given $\frac{dy}{dx} = \frac{1}{x^2y^3+xy}$

$$\frac{dx}{dy} = x^2y^3 + xy$$

$$\frac{dx}{dy} - xy = x^2y^3$$

Divide by $x^2 \rightarrow \frac{1}{x^2}\frac{dx}{dy} - \frac{1}{x^2}xy = \frac{1}{x^2}x^2y^3$

$\Rightarrow x^{-2}\frac{dx}{dy} - yx^{-1} = y^3$

let $x^{-1} = z$

$-1x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$

$x^{-2}\frac{dx}{dy} = -1\frac{dz}{dy}$

$\Rightarrow -1\frac{dz}{dy} - yz = y^3$

$\Rightarrow \frac{dz}{dy} + yz = -y^3 \Rightarrow \left[\frac{dz}{dy} + Pz = Q \therefore \text{linear differential equation in } y\right]$.

where $P = y$ $Q = -y^3$

Integrating factor : $e^{\int P dy} = e^{\int y dy} = e^{\frac{y^2}{2}}$

$\therefore$ General Solution: $z(I\cdot F) = \int Q(IF)\,dy + c$

$z \cdot e^{y^2/2} = \int -y^3 e^{y^2/2} dy + c$.

$z \cdot e^{y^2/2} = -\int y^3 e^{y^2/2} dy + c$

$z \cdot e^{y^2/2} = -\int y \cdot y^2 e^{y^2/2} dy + c$

let $\frac{y^2}{2} = t$. let $\frac{y^2}{2} = t$

$\cancel{2}ydy \cancel{2}dt$ $\cancel{/}ydy = \cancel{2}tdt$

$ydy = dt$. $ydy = tdt$.

$= -\int 2te^t dt + c$

$z \cdot e^{y^2/2} = -\left[2te^t - (1)e^t\right] + c$

$\frac{-1}{x} e^{y^2/2} = \frac{2}{}[xte^t - (1)e^t] + c$

$\frac{e^{y^2/2}}{x} = 2[e^t - te^t + c]$

$\frac{e^{y^2/2}}{x} = 2\{e^{y^2/2} - \frac{y^2}{2} e^{y^2/2}\} + c$

Q4. Solve $\frac{dy}{dx} + y\tan x = y^2 \sec x$.

Sol: Given $\frac{dy}{dx} + y\tan x = y^2\sec x$.

Divide throughout by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{y\tan x}{y^2} = \frac{y^2\sec x}{y^2}$$

$$y^{-2}\frac{dy}{dx} + \frac{\tan x}{y} = \sec x$$

$$y^{-2}\frac{dy}{dx} + \tan x \cdot y^{-1} = \sec x$$

Let $y^{-1} = z$

$$-1y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$y^{-2}\frac{dy}{dx} = -1\frac{dz}{dx}$$

$$-1\frac{dz}{dx} + \tan x \cdot z = \sec x$$

$$\Rightarrow \boxed{\frac{dz}{dx} - \tan x \cdot z = -\sec x}$$

$\left[\frac{dz}{dx} + Pz = Q(x) \Rightarrow\right.$ linear differential eq^n in x].

where $P = -\tan x$, $Q = -\sec x$

Integrating factor : $e^{\int P dx} = e^{\int -\tan x} = e^{-\int \tan x}$

$= e^{-\log|\sec x|} = \sec^{-1} x = \frac{1}{\sec x}$

$= \cos x$

$\therefore$ General Solution:

$$\boxed{z(I.F) = \int Q(I.F)\,dx + c}$$

$$z \cdot \cos x = \int -\sec x \cos x \, dx + c$$

$$y^{-1} \cos x = \int -dx + c$$

$$y^{-1} \cos x = -x + c$$

$$\frac{\cos x}{y} = -x + c$$

$$\cos x = -xy + c$$

$$\cos x + xy = c \,//.$$

Vimp 7 Marks.

(4) Recathis Differential Equation:

→ An equation of the type $\boxed{y' = Py^2 + Qy + R}$ is called Recathis differential equation.

→ If the solution is given $y = v$

→ General Solution is obtained by ~~y = x~~ ~~y = x + v~~

$$\boxed{y = v + \frac{1}{z}}$$

1Q. Find the general Solution of differential equation $y' = y^2 - (2x-1)y + x^2 - x + 1$ if the solution is given by $y = x$.

Vimp

Sol: Given $y' = y^2 - (2x-1)y + x^2 - x + 1$ —— (1) $y = x$ is solution

$y = x + \frac{1}{z}$ be the solution of (1)

$y' = 1 - \frac{1}{z^2}\frac{dz}{dx}$ Substitute in (1).

$$1 - \frac{1}{z^2}\frac{dz}{dx} = \left(x + \frac{1}{z}\right)^2 - (2x-1)\left(x + \frac{1}{z}\right) + x^2 - x + 1$$

$$1 - \frac{1}{z^2}\frac{dz}{dx} = x^2 + \frac{1}{z^2} + 2x\frac{1}{z} - 2x^2 - 2x\frac{1}{z} + x + \frac{1}{z} + x^2 - x + 1$$

$$-\frac{1}{z^2}\frac{dz}{dx} = \frac{1}{z^2} + \frac{1}{z}$$

$$\frac{-1}{\cancel{z^2}}\frac{dz}{dx} = \frac{1+z}{\cancel{z^2}}$$

$$\frac{dz}{dx} = -1 - z$$

$$\boxed{\frac{dz}{dx} + z = -1}$$

where $P = 1$, $Q = -1$.

Integrating factor: $e^{\int P dx} = e^{\int 1 dx}$
$= e^x$.

$\therefore$ General Solution $\cancel{f(I \cdot F)} = Q\int(I\cdot F)\,dx$

$$\boxed{z(I\cdot F) = \int Q(IF)\,dx + c}$$

$$\Rightarrow ze^{x} = \int -1\cdot e^{x}\,dx + c$$

$$\Rightarrow ze^{x} = \int -e^{x}\,dx + c$$

$$\Rightarrow ze^{x} = -e^{x} + c$$

$$\Rightarrow ze^{x} + e^{x} = c$$

$$\cancel{\Rightarrow z = \frac{-e^{x}}{e^{x}} + \frac{c}{e^{x}}}$$

$$\Rightarrow z = ce^{-x} - 1$$

Substitute in (2).

General Solution $\boxed{y = x + \frac{1}{z}}$

$$y = x + \frac{1}{ce^{-x} - 1}.$$

Imp.

Q2. Find the general solution of differential equation $y' = 2xy^2 + [1-4x]y + 2x - 1$ if $y = 1$ be the solution.

Sol. Given $y' = 2xy^2 + [1-4x]y + 2x - 1$ — (1).

$y = 1$ is a solution of (1) — (2).

let $\cancel{y'}$ $y = 1 + \frac{1}{z}$ be the solution of (1) — (2).

Substitute in (1).

$$y' = 0 - \frac{1}{z^2}\frac{dz}{dx} \quad \text{Substitute in (1)}$$

$$0 - \frac{1}{z^2}\frac{dz}{dx} = 2x\left(1 + \frac{1}{z}\right)^2 + [1-4x]\left(\frac{1+1}{z}\right) + 2x - 1$$

$$0 - \frac{1}{z^2}\frac{dz}{dx} = 2x\left(1 + \frac{1}{z^2} + \frac{2}{z}\right) + [1-4x]\left[1 + \frac{1}{z}\right] + 2x - 1$$

$$= \cancel{2/x} + \frac{2x}{z^2} + \cancel{\frac{2x}{z}} + \cancel{x} - \cancel{4/x} + \frac{1}{z} - \cancel{\frac{4x}{z}} + \cancel{2/x} - \cancel{x}$$

$$= \frac{2x}{z^2} + \frac{1}{z}$$

$$\therefore -\frac{1}{z^2}\frac{dz}{dx} = \frac{2x}{z^2} + \frac{1}{z}$$

$$\frac{-1}{\cancel{z^2}}\frac{dz}{dx} = \frac{2x+z}{\cancel{z^2}}$$

$$\frac{dz}{dx} = -2x+z$$

$$\boxed{\frac{dz}{dx} + z = -2x} \Rightarrow \frac{dz}{dx} + Pz = Q \text{ linear differential eq}^n \text{ in } x.$$

where $P = 1$, $Q = -2x$.

Integrating factor $= e^{\int P dx} = e^{\int 1 dx} = e^{[x]} = e^x$.

$\therefore$ General Solution $= \boxed{z(I\cdot F) = \int Q(I\cdot F)\,dx + C}$

$$z(e^x) = \int -2x\cdot e^x\,dx + C$$

$$ze^x = -2\int xe^x\,dx + C$$

$$ze^x = -2\left[xe^x - e^x\right] + C$$

$$z\cancel{e^x} = \cancel{e^x}\left[-2x+2\right] + C$$

$$z = [2-2x] + C$$

$\therefore$ The Required Solution is $y = 1 + \frac{1}{z}$.

$$\Rightarrow y = 1 + \frac{1}{2-2x+C}$$

//.

Q- Solve $\frac{dy}{dx} = 4x^2[y-x^2]^2 + \frac{y}{x}$ where $y = x$ is a solution.

Sol: Given $\frac{dy}{dx} = 4x^2[y-x^2]^2 + \frac{y}{x}$ —— ①

given $y = x$ is a solution of ①.

$\boxed{y = x + \frac{1}{z}}$ is a solution of ① —— ②.

$y' = 1 - \frac{1}{z^2}\frac{dz}{dx}$; Substitute in ①.

$1 - \frac{1}{z^2}\frac{dz}{dx} = 4x^2[x-x^2]\cdot 4x^2\left[\cancel{x} + \frac{1}{z} - \cancel{x^2}\right]^2 + \frac{\left[x+\frac{1}{z}\right]}{x}$

$1 - \frac{1}{z^2}\frac{dz}{dx} = 4x^{\cancel{3}} + \frac{4x^2}{z^2} - 4\cancel{x^4} + \frac{x}{x} + \frac{1}{zx}$

$= \frac{4x^2}{z^2} + 1 + \frac{x^1}{zx}$

$\cancel{1} - \frac{1}{z^2}\frac{dz}{dx} = \frac{4x^2}{z^2} + \cancel{1} + \frac{1}{zx}$

P.T.O $\longrightarrow$

$\frac{-1}{z^2}\frac{dz}{dx} = \frac{4x^2 + zx}{z^2}$

$-1\frac{dz}{dx} = 4x^2 + zx$

$\frac{dz}{dx} = -4x^2$ X

→ Given $\frac{dy}{dx} = 4x^2[y-x]^2 + \frac{y}{x}$ —— ①

given $y=x$ is a solution of ①.

$y = x + \frac{1}{z}$ is a solution of ① —— ②

$y' = 1 - \frac{1}{z^2}\frac{dz}{dx}$; Substitute in ①.

$1 - \frac{1}{z^2}\frac{dz}{dx} = 4x^2 \left[\cancel{x} + \frac{1}{z} - \cancel{x} \right]^2 + \frac{x + \frac{1}{z}}{x}$

$= 4x^2 \left[\frac{1}{z} \right]^2 + \frac{x}{x} + \frac{1}{xz}$

$\cancel{1} - \frac{1}{z^2}\frac{dz}{dx} = \frac{4x^2}{z^2} + \cancel{1} + \frac{1}{xz}$

$z^2\left(\frac{-1}{z^2}\frac{dz}{dx}\right) = z^2\left(\frac{4x^2}{z^2} + \frac{1}{zx}\right) \Rightarrow \frac{dz}{dx} = -4x^2 - \frac{z}{x}$.

$\frac{dz}{dx} = -4x^2 - \frac{z}{x}$

$\boxed{\frac{dz}{dx} + \frac{z}{x} = -4x^2}$ ⇒ This is of the form $\frac{dz}{dx} + Pz = Q$

$I.F = e^{\int P dx} = e^{\int \frac{1}{x} dx}$

$= e^{\log x} = x$

where $P = \frac{1}{x}$, $Q = -4x^2$

∴ General Solution : $\boxed{z(I.F) = \int Q(I.F)\,dx + c}$

$z \cdot x = \int -4x^2 \cdot x + c$

$z \cdot x = -4\int x^3 dx + c$

$zx = -\cancel{4}\frac{x^4}{\cancel{4}} + c$

$zx + x^4 = c$

$z = \frac{-x^{\cancel{4}}}{\cancel{x}} + \frac{c}{x}$

$z = -x^3 + cx^{-1}$

Substitute $z = (\cdot cx^{-1} - x^3)$ in ② we get,

Required solution is $y = x + \frac{1}{z}$.

$$\boxed{y = x + \frac{1}{cx^{-1} - x^3}}$$

//

24/06/2025 ⑤ Clairaits Differential Equation.

An ~~eqy~~ equation of the type $\boxed{y = p(x) + f(p)}$ where $p = \frac{dy}{dx}$ is called Clairits differential equation.

Its general solution is $\boxed{cx + f(c) = y}$

1Q. Solve $y = px + p^2$.

Sol. $y = px + p^2$ — ①

① is clairits differential equation.

$y = cx + c^2$ — ②

Differentiate partially to c.

$y - cx - c^2 = 0$

$0 - x - 2c = 0$

$-x = 2c$

$\boxed{c = \frac{-x}{2}}$ — ③.

Substitute in ②

$y = -\frac{x}{2}(x) + \left(-\frac{x}{2}\right)^2$

$y = \frac{-x^2}{2} + \frac{x^4}{4} = \frac{-x^2}{4}$

$y = \frac{-x^2}{4}$ is the solution.

$y + \frac{x^2}{4} = 0$.

2Q. Solve $y = px - p^3$.

Sol: Given $y = px - p^3$ —— ①

① is clairauts differential equation.

$y = cx - c^3$ —— ②

Differentiate partially to 'c'

$y - cx + c^3 = 0$

$0 - x + 3c^2 = 0$

$+x = 3c^2$

$3c^2 = +x$

$c^2 = \frac{x}{3}$

$\boxed{c = \sqrt{\frac{x}{3}}}$

Substitute in ② we get;

$y = \sqrt{\frac{x}{3}}(x) - \left(\sqrt{\frac{x}{3}}\right)^3$

$y = \sqrt{\frac{x}{3}}\, x - \left(\sqrt{\frac{x}{3}}\right)^3$ is the solution.

3Q. Solve $y = xy' + \sin^{-1}(y')$

Sol. Given $y = xy' + \sin^{-1}(y')$ —— ①

$\boxed{p = \frac{dy}{dx} = y'}$

① is Clairauts differential equation

$y = xp + \sin^{-1}(p)$

General solution is $y = cx + \sin^{-1}(c)$

$y - cx - \sin^{-1}(c) = 0$ —— ②

Partially differentiate w.r.t 'c'

$0 - x - \frac{1}{\sqrt{1-c^2}} = 0$

$x = \frac{-1}{\sqrt{1-c^2}}$

$\sqrt{1-c^2} = \frac{-1}{x}$ $\Rightarrow$ $1 - c^2 = \frac{1}{x^2}$ $\Rightarrow 1 - \frac{1}{x^2} = c^2$

$\boxed{c = \sqrt{1 - \frac{1}{x^2}}}$

$y - \left(\sqrt{\frac{x^2-1}{x}}\right)\cancel{x} - \sin^{-1}\left(\sqrt{\frac{x^2-1}{x}}\right)$

$y - \left(\sqrt{x^2-1}\right) - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$.

Substitute in ②.

$y = \frac{x}{\sqrt{1-\frac{1}{x^2}}}$. $y = \frac{x}{\sqrt{1-x^2}} - \sin^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$.

⑥ Orthogonal Trajectories. x axis ⟷ y axis

$\sin x \longleftrightarrow \cos x$

Working Rule:

① Given differential equation $f(x, y, c) = 0 \longrightarrow ①$

② Differentiate with respect to x of eqⁿ ① — ② and eliminate arbitrary constant 'c' by using eqⁿ ② and ①.

③ Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ and solve integrate

Q1. Find orthogonal trajectory of $ay^2 = x^3$.

Sol. Given $ay^2 = x^3$ — ①

differentiate w.r.t 'x';

$a \cdot 2y \frac{dy}{dx} = 3x^2$ — ②

Dividing ① by ② to eliminate 'a'

$\frac{①}{②} = \frac{ay^2}{a \cdot 2y \frac{dy}{dx}} = \frac{x^3}{3x^2}$

$\Rightarrow \frac{y}{2\frac{dy}{dx}} = \frac{x}{3}$

$\boxed{3y = 2x \frac{dy}{dx}}$ Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$3y = 2x\left(-\frac{dx}{dy}\right)$

Q. Find orthogonal trajectory of $r = ce^{\theta}$ where 'c' is paramt.

Sol. $\log r = \log ce^{\theta}$

$\log r = \log c + \log e^{\theta}$

$\log r = \log c + \theta$

D.w.r.t $\theta \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = 0 + 1$

Replace $\frac{dr}{d\theta} \Rightarrow -r^2\frac{d\theta}{dr}$

$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = 1 \Rightarrow -r\frac{d\theta}{dr} = 1$

$\Rightarrow 3y\,dy = -2x\,dx$.

Integrate

$\int 3y\,dy = -2\int x\,dx$.

$3\frac{y^2}{2} = -\cancel{2}\cdot\frac{x^2}{\cancel{2}}$

$3\frac{y^2}{2} = -x^2 + c \longrightarrow \frac{3y^2}{2} + x^2 = c$ ✓

$x^2 = -\frac{3y^2}{2} + c$ is the required orthogonal trajectory.

*** Vimp

26. Find orthogonal trajectory of $x^2+y^2=ax$.

Sol: Given $x^2+y^2 = ax$ —(1)

differentiate w.r.t 'x';

$2x + 2y\frac{dy}{dx} = a$ —(2)

[If you have a as subject substitute a in (1) or if there is only 1 term in LHS then divide].

Substitute eqⁿ (2) in (1).

$\Rightarrow x^2+y^2 = \left(2x+2y\frac{dy}{dx}\right)x$

$\Rightarrow x^2+y^2 = 2x^2 + 2xy\frac{dy}{dx}$

$\Rightarrow x^2+y^2-2x^2-2xy\frac{dy}{dx} = 0$

$\Rightarrow \boxed{-x^2+y^2-2xy\frac{dy}{dx} = 0}$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$

$\Rightarrow -x^2+y^2-2xy\left(-\frac{dx}{dy}\right) = 0$

$\Rightarrow -x^2+y^2+2xy\frac{dx}{dy} = 0$

$\Rightarrow y^2dy - x^2dy + 2xy\,dx = 0$

divide by y^2

$\Rightarrow dy - \left[\frac{x^2dy - y\,2x\,dx}{y^2}\right] = 0$

$dy + \left[\frac{y\,2x\,dx - x^2dy}{y^2}\right] = 0$

$dy + d\left[\frac{x^2}{y}\right] = 0$

Integrate;

$$\int dy + \int d\left(\frac{x^2}{y}\right) = c$$

$$y + \frac{x^2}{y} = c$$

$\boxed{y^2 + x^2 = cy}$ is the required orthogonal trajectory.

Vimp

Q3. Find the orthogonal trajectory of $x^2+y^2+2gx+c=0$. where g is arbitrary constant.

Sol: Given $x^2+y^2+2gx+c=0$ — ①

Differentiate w.r.t x of ①

$$2x + 2y\frac{dy}{dx} + 2g = 0$$

$$g = -\left[x + y\frac{dy}{dx}\right] \quad — ②$$

Substitute in ①.

$$\Rightarrow x^2+y^2+2\left[-\left(x+y\frac{dy}{dx}\right)x\right]+c=0$$

$$\Rightarrow x^2+y^2-2x^2-2xy\frac{dy}{dx}+c=0$$

$$\Rightarrow y^2-x^2-2xy\frac{dy}{dx}+c=0$$

Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

$$\Rightarrow y^2-x^2-2xy\left(-\frac{dx}{dy}\right)+c=0$$

$$\Rightarrow y^2-x^2+2xy\frac{dx}{dy}+c=0$$

$$\Rightarrow y^2dy - x^2dy + 2xy\,dx + c\,dy = 0$$

$\div$ by $y^2 \Rightarrow \dfrac{y^2dy - y2x\,dx - x^2dy}{y^2} + \dfrac{c}{y^2}dy = 0$

$$dy + \frac{y\,2x\,dx - x^2 dy}{y^2} + \frac{c}{y^2}\,dy = 0$$

$$dy + d\left(\frac{x^2}{y}\right) + c\,y^{-2}dy = 0$$

Integrate

$$\int dy + \int d\left(\frac{x^2}{y}\right) + c\int y^{-2}dy = k$$

$$y + \frac{x^2}{y} + c\,\frac{y^{-1}}{-1} = k$$

$$y + \frac{x^2}{y} - \frac{c}{y} = k$$

$$\frac{y^2 + x^2 - c}{y} = k$$

$$y^2 + x^2 - c = ky$$

$\boxed{y^2 + x^2 - ky - c = 0}$ is the required orthogonal trajectory.

Q. Find orthogonal trajectory of $a(1-\cos\theta) = r$.

Sol: $r = a(1-\cos\theta)$ Take log on b-s,

$\log r = \log a(1-\cos\theta)$

$\log r = \log a + \log(1-\cos\theta)$ — ①.

Diff. w.r.t. 'θ' we get;

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}\cdot\sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\theta/2\cos\theta/2}{2(\sin\theta/2)^2} \Rightarrow \cot\theta/2$$

7 Marks -

⑦ Self Orthogonal.

If a curve ~~intercept~~ intersect itself ~~with~~ orthogonally then it is said to be self orthogonal.

1Q. Show that the family of $y^2 = 4a[x+a]$ is self orthogonal.

Sol: given $y^2 = 4a[x+a]$ — ①

Differentiate with respect to x of ①.

$$2y\cdot\frac{dy}{dx} = 4a[1+0]$$

$$\cancel{2}yy' = \cancel{4}^{2}a$$

$\boxed{\frac{yy'}{2} = a}$ — ② substitute in ①.

$$y^2 = 4\left[\frac{yy'}{2}\right]\left[x + \frac{yy'}{2}\right].$$

$\rightarrow \quad y^2 = 4\left[\frac{yy'}{2}\right]\left[x+\frac{yy'}{2}\right]$

$y = 2y'\left[\frac{2x+yy'}{2}\right]$

$\boxed{y = y'[2x+yy'] \quad — (3)}$

Replace y' by $-\frac{1}{y'}$

$\boxed{\begin{aligned} y' &= \frac{dy}{dx} \\ -\frac{1}{y'} &= \frac{dx}{dy} \end{aligned}}$

$y = -\frac{1}{y'}\left[2x - y\frac{1}{y'}\right]$

$yy' = -\left[\frac{2xy'-y}{y'}\right]$

$yy'y' = -2xy' + y$

$yy'y' + 2xy' = y$

$\boxed{y = y'[2x+yy'] \quad — (4)}$

Before substituting y' and after substituting the answer must be same.

(3) = (4)

$\therefore$ It is self orthogonal.

14 Marks.

2Q. Show that family of ellipse $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is self orthogonal where λ is arbitrary constant.

Sol. Given $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ —— ①

Differentiate w.r. to x of ①.

$$\frac{2x}{a^2+\lambda} + \frac{2y^2y'}{b^2+\lambda} = 0$$

$$\frac{2xb^2+2x\lambda+2y^2y'a^2+2y^2y'\lambda}{(a^2+\lambda)(b^2+\lambda)} = 0$$

$$\frac{xb^2+x\lambda+y^2y'a^2+y^2y'\lambda}{(a^2+\lambda)(b^2+\lambda)} = 0$$

$$xb^2+yy'a^2+\lambda[x+yy'] = 0$$

$$-\lambda[x+yy'] = xb^2+yy'a^2$$

$$\lambda = -\frac{[xb^2+yy'a^2]}{x+yy'}$$

$\therefore$

$$a^2+\lambda = a^2 - \left[\frac{xb^2+yy'a^2}{x+yy'}\right]$$

$$a^2+\lambda = \frac{a^2x+a^2yy'-xb^2-a^2yy'}{x+yy'}$$

$$\boxed{a^2+\lambda = \frac{x[a^2-b^2]}{x+yy'}} \rightarrow ②$$

$$b^2+\lambda = b^2 - \frac{[xb^2 + yy'a^2]}{x+yy'}$$

$$b^2+\lambda = \frac{b^2x + b^2yy' - xb^2 - yy'a^2}{x+yy'}$$

$$= \frac{b^2yy' - a^2yy'}{x+yy'}$$

$$\boxed{b^2+\lambda = \frac{yy'(b^2-a^2)}{x+yy'}} \rightarrow (3)$$

Substitute in (1).

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$$

$$\frac{x^2}{\frac{x(a^2-b^2)}{x+yy'}} + \frac{y^2}{\frac{yy'(b^2-a^2)}{x+yy'}} = 1$$

$$\frac{x(x+yy')}{a^2-b^2} + \frac{y(x+yy')}{y'(b^2-a^2)} = 1.$$

$$\frac{x(x+yy')}{a^2-b^2} - \frac{y(x+yy')}{y'(a^2-b^2)} = 1$$

$$\frac{xy'(x+yy') - y(x+yy')}{y'(a^2-b^2)} = 1$$

$$\frac{x^2y' + xyy'y' - xy - y^2y'}{y'(a^2-b^2)} = 1$$

$$x^2y'+xyy'y'-xy-y^2y'=y'(a^2-b^2)$$

$$x^2y'+xyy'y'-xy-y^2y'-y'(a^2-b^2)=0$$

$$y'(x^2+xyy')-y(x-yy')-y'(a^2-b^2)=0 \longrightarrow (4)$$

Replace y' by $-\frac{1}{y'}$

$$-\frac{1}{y'}\left[x^2+xy\left(-\frac{1}{y'}\right)-y^2\right]-xy=-\frac{1}{y'}\left[a^2-b^2\right]$$

$$x^2-xy\frac{1}{y'}-y+xyy'=\left[a^2-b^2\right]$$

$$x^2y'-xy-yy'+xyy'y'=y'\left[a^2-b^2\right]$$

$$y'\left[x^2+xyy'-\underline{4}\right]-xy=y'\left[a^2-b^2\right] \longrightarrow (5)$$

(4) = (5)

$\therefore$ It is self orthogonal.