

SCALAR AND VECTOR FIELD

SCALAR FIELD:

A Point function that assigns a scalar to every point in some region in the plane or in space is called a scalar field.

Eg: Temperature distribution throughout the body.

VECTOR FIELD:

A point function that assigns a vector to every point in some region in plane or in space is called a vector field.

Eg: Electric field and magnetic field

VECTOR DIFFERENTIAL OPERATOR:

The Operator " ∇ " (Del) is defined as $\nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z}$

" ∇ " is called a vector differential operators, it involves the unit vector i, j, k and also the partial derivatives

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}.$$

GRADIENT:

Let $\phi(x, y, z)$ be differential at each point in a certain region, then Gradient of ϕ , it is written as

$$\nabla \phi. \text{ i.e. } \nabla \phi = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \phi$$
$$= i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}.$$

$$\nabla \phi = \text{grad } \phi.$$

UNIT NORMAL TO SURFACE: Let $\phi(x, y, z) = C$, then
 $n = \frac{\nabla \phi}{|\nabla \phi|}$ is unit Normal to surface at a point

$p(x, y, z)$.

ANGLE BETWEEN TWO SURFACES:

Let $\phi_1(x, y, z) = C_1$ and $\phi_2(x, y, z) = C_2$ be the two surfaces at a common point p , the angle between the two surfaces is the angle between the two normals.

$$\cos \theta = \frac{(\nabla \phi_1)(\nabla \phi_2)}{|\nabla \phi_1| |\nabla \phi_2|}$$

$$\theta = \cos^{-1} \left[\frac{(\nabla \phi_1)(\nabla \phi_2)}{|\nabla \phi_1| |\nabla \phi_2|} \right]$$

DIRECTIONAL DERIVATIVE:

The directional derivative of $F(x, y, z)$ in direction of $\vec{a}$ vector is given by $\Delta F \frac{(\vec{a})}{|\vec{a}|}$.

Also the directional derivative of F at a point p is maximum in the direction of Normal at p and its maximum value = $|\nabla \phi|$.

Q1: Find Unit Normal to Surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$.

Sol: Let $\phi(x, y, z) = x^3 + y^3 + 3xyz - 3$, $P(1, 2, -1)$
Unit Normal to the surface $\phi(x, y, z)$; $\eta = \frac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^3 + y^3 + 3xyz - 3)$$

$$= i(3x^2 + 3yz) + j(3y^2 + 3xz) + k(3xy)$$

$$\nabla\phi_{(1,2,-1)} = i(3 - 6) + j(12 - 3) + k(6)$$

$$= -3i + 9j + 6k$$

$$|\nabla\phi| = \sqrt{9 + 81 + 36} = 3\sqrt{14} = \sqrt{12 \times 8} = \sqrt{9 \times 14}$$

$$\eta = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3i + 9j + 6k}{3\sqrt{14}} = \frac{-i + 3j + 2k}{\sqrt{14}}$$

Q2: If $\phi(x, y, z) = 3x^2y - y^3z^3$ at a point $P(1, -2, -1)$. Find grad ϕ .

Sol:

$$\phi(x, y, z) = 3x^2y - y^3z^3$$

$$\text{grad } \phi = \nabla\phi$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (3x^2y - y^3z^3)$$

$$= i(6xy) + j(3x^2 - 3y^2z^3) + k(-3y^3z^2)$$

$$\text{grad } \phi_{(1,-2,-1)} = (6)(1)(-2)i + j(3 - 3(4)(-1)) + 3(-1)^2k$$

$$\text{grad } \phi_{(1,-2,-1)} = -12i + 15j + 24k$$

Q4: If θ is the acute angle b/w the surfaces $xy^2z = 3xz^2$ and $3x^2 + 2z - y^2 = 1$ at pt $(1, -2, 1)$. So $\cos \theta = 3/2\sqrt{6}$.

Sol: $\phi_1(x, y, z) = xy^2z - 3xz^2$, $\phi_2(x, y, z) = 3x^2 + 2z - y^2$
at a pt $(1, -2, 1)$

The angle between surfaces ϕ_1 & ϕ_2 is

$$\cos \theta = \frac{(\nabla \phi_1) \cdot (\nabla \phi_2)}{|\nabla \phi_1| |\nabla \phi_2|}$$

$$\begin{aligned} \nabla \phi_1 &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (xy^2z - 3xz^2) \\ &= i(zy^2 - 3z) + j(2xyz) + k(xy^2 - 2z) \end{aligned}$$

$$\begin{aligned} \nabla \phi_1(1, -2, 1) &= i(4 - 3) + j(-4) + k(4 - 2) \\ &= i - 4j + 2k \end{aligned}$$

$$|\nabla \phi_1| = \sqrt{1 + 16 + 4} = \sqrt{21}$$

$$\begin{aligned} \nabla \phi_2 &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (3x^2 + 2z - y^2) \\ &= i(6x) + j(-2y) + k(2) = 6xi - 2yj + 2k \end{aligned}$$

$$\nabla \phi_2(1, -2, 1) = 6i + 4j + 2k$$

$$|\nabla \phi_2| = \sqrt{36 + 16 + 4} = \sqrt{56}$$

$$\cos \theta = \frac{(\nabla \phi_1) \cdot (\nabla \phi_2)}{|\nabla \phi_1| |\nabla \phi_2|} = \frac{(i - 4j + 2k) \cdot (6i + 4j + 2k)}{\sqrt{21} \times \sqrt{56}}$$

Q3: Find the Angle b/w two Surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2+3$ at $(2, -1, 2)$.

Sol: Let $\phi_1(x, y, z) = x^2 + y^2 + z^2 - 9$

$$\phi_2(x, y, z) = z - x^2 - y^2 + 3$$

Angle b/w two Surfaces is the angle between the two normals i.e. $\theta = \cos^{-1} \left[\frac{(\nabla \phi_1) \cdot (\nabla \phi_2)}{|\nabla \phi_1| |\nabla \phi_2|} \right]$

$$\begin{aligned} \nabla \phi_1 &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 9) \\ &= 2(x i + y j + z k) \end{aligned}$$

$$\nabla \phi_1(2, -1, 2) = 2(2i - j + 2k) = 4i - 2j + 4k.$$

$$|\nabla \phi_1| = \sqrt{16 + 4 + 16} = 6.$$

$$\begin{aligned} \nabla \phi_2 &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (z - x^2 - y^2 + 3) \\ &= -2xi - 2yj + k \end{aligned}$$

$$\nabla \phi_2(2, -1, 2) = (-2)(2)i - (2)(-1)j + k = -4i + 2j + k$$

$$|\nabla \phi_2| = \sqrt{16 + 4 + 1} = \sqrt{21}.$$

$$\theta = \cos^{-1} \left[\frac{(4i - 2j + 4k) \cdot (-4i + 2j + k)}{6\sqrt{21}} \right]$$

$$\theta = \cos^{-1} \left(\frac{-16 - 4 + 4}{6\sqrt{21}} \right) = \cos^{-1} \left(\frac{-8}{3\sqrt{21}} \right)$$

$$\theta = 54.41^\circ.$$

4 Ans
Cont

$$= \frac{6-16+4}{\sqrt{3 \times 7 \times 8 \times 7}} = \frac{-6}{7 \times 2\sqrt{6}} = \frac{-3}{7\sqrt{6}}$$

$$\cos \theta = \frac{3}{7\sqrt{6}}$$

Q5: In what direction $(3, 1, -2)$ is directional derivative of $\phi = x^2 + y^2 + z^2$ maximum. Find the magnitude of this maximum.

Sol: The directional derivative if N is the normal at the point $(3, 1, -2)$ then $N = |\nabla \phi|_{(3, 1, -2)}$

$$\begin{aligned} \nabla \phi &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^2 y^2 z^4) \\ &= i (2xy^2z^4) + j (2x^2yz^4) + k (4x^2y^2z^3) \\ &= i (2 \times 3 \times 1 \times 16) + j (2 \times 9 \times 16) + k (4 \times 9 \times 8) \\ &= 2 \times 3 \times 16 (i + 3j - 3k) \end{aligned}$$

$$\nabla \phi = 96(i + 3j - 3k), |\nabla \phi| = 96\sqrt{1+9+9} = 96\sqrt{19}$$

$\therefore$ The MAX. VALUE = $96\sqrt{19}$.

Q6: Find the directional derivative of $f(x, y, z) = 4e^{2x-y+z}$ at pt $(1, 1, -1)$ in the direction towards the point $(-3, 5, 6)$.

Sol: Given,

$$\vec{a} = -3i + 5j + 6k$$

$$|\vec{a}| = \sqrt{9+25+36} = \sqrt{70}$$

The directional derivative of f in the direction of

$$\vec{a} \text{ is } \frac{\vec{a}}{|\vec{a}|}$$

$$\nabla f = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (4e^{2x-y+z})$$

$$= i 4e^{2x-y+2z} (2) + j \cdot 4e^{2x-y+2z} (-1) + k 4e^{2x-y+2z} (1)$$

$$\nabla f_{(1,1,1)} = 4e^{2-1-1} [2i - j + k]$$

$$\nabla f = 4(2i - j + k)$$

$$\begin{aligned} \text{then Directional Derivative} &= \nabla f \cdot \frac{\vec{a}}{|\vec{a}|} \\ &= 4(2i - j + k) \cdot \frac{(-3i + 5j + 6k)}{\sqrt{70}} \\ &= 4 \frac{(-6 - 5 + 6)}{\sqrt{70}} = \frac{-20}{\sqrt{70}} \end{aligned}$$

Q7: Find Directional Derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, 1)$ in direction $2i - j - 2k$.

Sol: $\vec{a} = 2i - j - 2k$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\begin{aligned} \nabla \phi &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^2yz + 4xz^2) \\ &= i(2xyz + 4z^2) + j(x^2z) + k(x^2y + 8xz) \end{aligned}$$

$$\nabla \phi_{(1,-2,1)} = i(2(1)(-2)(1) + 4(1)^2) + j(1 \cdot 1) + k(1(-2) + 8)$$

$$\nabla \phi = -j + 6k$$

$$D.O = \nabla \phi \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{(-j + 6k)(2i - j - 2k)}{3}$$

$$= \frac{-1-12}{3} = \frac{-13}{3}$$

Q8: Find D.D of $\phi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in direction $i + 2j + 2k$.

Sol: $\vec{a} = i + 2j + 2k$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\begin{aligned}\nabla\phi &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy^2 + yz^3) \\ &= y^2i + j(2xy + z^3) + k(3yz^2)\end{aligned}$$

$$\begin{aligned}\nabla\phi_{(2,-1,1)} &= i + j(-4+1) + 3k(1(-1)) \\ &= i - 3j - 3k\end{aligned}$$

$$D.D = \frac{(i - 3j - 3k)(i + 2j + 2k)}{3} = \frac{1 - 6 - 6}{3} = -11/3$$

Q9: Find D.D of $\phi = x^2 - y^2 + 2xz^2$ at pt $(1, 2, 3)$ in direction of line PQ where $P(1, 2, 3)$ & $Q(5, 0, 4)$.

Sol: $\vec{a} = \vec{PQ} = \vec{OQ} - \vec{OP} = (5i + 0j + 4k) - (1i + 2j + 3k)$

$$\vec{a} = 4i - 2j + k, \quad |\vec{a}| = \sqrt{16+4+1} = \sqrt{21}$$

$$\begin{aligned}\nabla\phi &= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 - y^2 + 2xz^2) \\ &= i(2x) + j(-2y) + k(4z)\end{aligned}$$

$$\nabla\phi_{(1,2,3)} = 2i - 4j + 12k$$

$$D.D = \frac{(2i - 4j + 12k)(4i - 2j + k)}{\sqrt{16+4+1}} = \frac{8+8+12}{\sqrt{21}}$$

$$D.D = \frac{28}{\sqrt{21}}$$

DIVERGENCE, CURL OF VECTOR FIELD

DIVERGENCE: If $V(x, y, z)$ is a continuously differentiable vector form, is a divergence curve of vector field.

$$\nabla \cdot V = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \cdot [V_1 i + V_2 j + V_3 k]$$
$$= \frac{\partial}{\partial x} V_1 + \frac{\partial}{\partial y} V_2 + \frac{\partial}{\partial z} V_3.$$

The scalar quantity $\nabla \cdot V$ is known as divergence of vector field 'V' and is denoted as 'div V'.

Note: A vector point function 'V' is said to be solenoidal if $\text{div } V = 0$.

SOME PROPERTIES OF DIVERGENCE:

The divergence of constant vector is zero.

The, $\text{div}(V_1 + V_2) = \text{div } V_1 + \text{div } V_2$, where

$$V_1 = V_1(x, y, z)$$

$$V_2 = V_2(x, y, z)$$

If $\phi = \phi(x, y, z)$ is a scalar point function and

$V = V(x, y, z)$ is a vector point function, then

$$\text{div}(V\phi) = \phi \text{div } V + V \cdot \text{grad } \phi.$$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then s.t. $\nabla r^n = n r^{n-1} \vec{r}$.

Given,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2x = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2y = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}} \cdot 2z = \frac{z}{r}$$

$$\nabla \cdot r^n = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] r^n$$

$$= \hat{i} \frac{\partial r^n}{\partial x} + \hat{j} \frac{\partial r^n}{\partial y} + \hat{k} \frac{\partial r^n}{\partial z} = \left[\hat{i} \cdot n \cdot r^{n-1} \left(\frac{\partial r}{\partial x} \right) + \hat{j} \cdot n \cdot r^{n-1} \left(\frac{\partial r}{\partial y} \right) + \hat{k} \cdot n \cdot r^{n-1} \left(\frac{\partial r}{\partial z} \right) \right]$$

$$= n \cdot r^{n-1} \left[\hat{i} \frac{\partial r}{\partial x} + \hat{j} \frac{\partial r}{\partial y} + \hat{k} \frac{\partial r}{\partial z} \right]$$

$$= n \cdot r^{n-1} \left[\frac{\hat{i}x}{r} + \frac{\hat{j}y}{r} + \frac{\hat{k}z}{r} \right] = n \frac{r^{n-1}}{r} (x\hat{i} + y\hat{j} + z\hat{k})$$

$$= n \cdot r^{n-2} \vec{r}$$

Prove that $\nabla \cdot r^n = n(n+1)r^{n-2}$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}$$

$$\nabla^2 r^n = \nabla \cdot \nabla r^n = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] r^n$$

$$= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] r^n$$

$$\text{Consider } \frac{\partial^2}{\partial x^2} (r^n) = \frac{\partial}{\partial x} \left(\frac{\partial}{\partial x} r^n \right)$$

$$= \frac{\partial}{\partial x} \left(n \cdot r^{n-1} \frac{\partial r}{\partial x} \right)$$

$$= \frac{\partial}{\partial x} \left(n \cdot r^{n-1} \cdot \frac{x}{r} \right) = \frac{\partial}{\partial x} \left(\frac{n \cdot r^{n-2} \cdot x}{1} \right)$$

$$= n(n-2) r^{n-3} \frac{\partial r}{\partial x} \cdot x + n \cdot r^{n-2}$$

$$= n(n-2) r^{n-3} \frac{x}{r} + n \cdot r^{n-2}$$

$$\therefore \frac{\partial^2 r^n}{\partial x^2} = n(n-2) r^{n-4} x^2 + n r^{n-2} \rightarrow \textcircled{1}$$

$$\text{Ily } \frac{\partial^2 r^n}{\partial y^2} = n(n-2) r^{n-4} y^2 + n r^{n-2} \rightarrow \textcircled{2}$$

$$\frac{\partial^2 r^n}{\partial z^2} = n(n-2) r^{n-4} z^2 + n r^{n-2} \rightarrow \textcircled{3}$$

$$\begin{aligned}
 \nabla^2 r^n &= n(n-2)r^{n-4}(x^2+y^2+z^2) + nr^{n-2} + n(n-2)r^{n-4}y^2 + nr^{n-2} + n(n-2)r^{n-4}z^2 + nr^{n-2} \\
 &= n(n-2)r^{n-4}(x^2+y^2+z^2) + 3nr^{n-2} \\
 &= n(n-2)r^{n-4}r^2 + 3nr^{n-2} \\
 &= n(n-2)r^{n-2} + 3nr^{n-2} = nr^{n-2}(n-2+3) \\
 \nabla^2 r^n &= n(n+1)r^{n-2}
 \end{aligned}$$

Q3: Prove that $\operatorname{div} \frac{\vec{r}}{r} = \frac{2}{r}$.

Sol: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
 $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\begin{aligned}
 \operatorname{div} \frac{\vec{r}}{r} &= \nabla \cdot \frac{\vec{r}}{r} \\
 &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \left(\frac{x\hat{i}}{r} + \frac{y\hat{j}}{r} + \frac{z\hat{k}}{r} \right) \\
 &= \frac{\partial}{\partial x} \left(\frac{x}{r} \right) + \left(\frac{\partial}{\partial y} \cdot \frac{y}{r} \right) + \left(\frac{\partial}{\partial z} \cdot \frac{z}{r} \right)
 \end{aligned}$$

Consider $\frac{\partial}{\partial x} \left(\frac{x}{r} \right) = \frac{x - \frac{\partial r}{\partial x} \cdot x}{r^2} = \frac{x}{r^2} - \frac{\partial r}{\partial x} \frac{x}{r}$

$$= \frac{1}{r} - \frac{x^2}{r^3} \longrightarrow \textcircled{1}$$

Similarly, $\frac{\partial}{\partial y} \left(\frac{y}{r} \right) = \frac{1}{r} - \frac{y^2}{r^3} \longrightarrow \textcircled{2}$

$$\frac{\partial}{\partial z} \left(\frac{z}{r} \right) = \frac{1}{r} - \frac{z^2}{r^3} \longrightarrow \textcircled{3}$$

$$\frac{\partial}{\partial x} \cdot \frac{x}{r} + \frac{\partial}{\partial y} \cdot \frac{y}{r} + \frac{\partial}{\partial z} \cdot \frac{z}{r} = \frac{3}{r} - \frac{(x^2 + y^2 + z^2)}{r^3}$$

$$= \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\operatorname{div} \frac{\vec{r}}{r} = \frac{2}{r}$$

Q4: Prove that $\operatorname{div}(r^n \vec{r}) = (n+3)r^n$

Sol:

$$\vec{r} = xi + yj + zk$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\operatorname{div} r^n \vec{r} = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] r^n (xi + yj + zk)$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (r^n xi + r^n yj + r^n zk)$$

$$= \frac{\partial}{\partial x} (r^n x) + \frac{\partial}{\partial y} (r^n y) + \frac{\partial}{\partial z} (r^n z)$$

Consider $\frac{\partial}{\partial x} (r^n x) = n r^{n-1} \frac{\partial r}{\partial x} x + (1)(r^n)$

$$= n r^{n-1} \frac{x}{r} \cdot x + r^n$$

$$\frac{\partial}{\partial x} (r^n x) = n r^{n-2} x^2 + r^n \longrightarrow (1)$$

$$\frac{\partial}{\partial y} (r^n y) = n r^{n-2} y^2 + r^n \longrightarrow (2)$$

$$\frac{\partial}{\partial z} (r^n z) = n r^{n-2} z^2 + r^n \longrightarrow (3)$$

$$\frac{\partial}{\partial x} (r^n x) + \frac{\partial}{\partial y} (r^n y) + \frac{\partial}{\partial z} (r^n z) = n r^{n-2} (x^2 + y^2 + z^2) + 3r^n$$

$$= n r^{n-2} r^2 + 3r^n$$

$$= n r^n + 3r^n = r^n (n+3)$$

$$\operatorname{div} r^n \vec{r} = (n+3)r^n$$

Q5: Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Sol:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\begin{aligned} \nabla^2 f(r) &= \nabla \cdot \nabla f(r) \\ &= \nabla \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) f(r) \\ &= \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r) \end{aligned}$$

$$\begin{aligned} \text{Consider } \frac{\partial^2}{\partial x^2} f(r) &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial x} f(r) \right] \\ &= \frac{\partial}{\partial x} \left(f'(r) \frac{\partial r}{\partial x} \right) = \frac{\partial}{\partial x} \left[f'(r) \cdot \frac{x}{r} \right] \\ &= f''(r) \frac{\partial r}{\partial x} \cdot \frac{x}{r} + f'(r) \left(\frac{1}{r} \right) - f'(r) \frac{x^2}{r^3} \end{aligned}$$

$$\frac{\partial^2}{\partial x^2} f(r) = f''(r) \frac{x^2}{r^2} + f'(r) \left(\frac{1}{r} \right) - f'(r) \frac{x^2}{r^3} \quad \text{--- (1)}$$

$$\text{Similarly, } \frac{\partial^2}{\partial y^2} f(r) = f''(r) \frac{y^2}{r^2} + \frac{f'(r)}{r} - f'(r) \frac{y^2}{r^3} \quad \text{--- (2)}$$

$$\frac{\partial^2}{\partial z^2} f(r) = f''(r) \frac{z^2}{r^2} + \frac{f'(r)}{r} - f'(r) \frac{z^2}{r^3} \quad \text{--- (3)}$$

Add (1), (2) & (3),

$$\begin{aligned} \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r) &= \left[\frac{f''(r)(x^2 + y^2 + z^2)}{r^2} - \frac{f'(r)}{r^3} (x^2 + y^2 + z^2) + \frac{3}{r} f'(r) \right] \\ &= f''(r) \frac{r^2}{r^2} + \frac{3}{r} f'(r) - \frac{f'(r) \cdot r^2}{r^3} \end{aligned}$$

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r).$$

Q6: Prove that $\text{div} \frac{\vec{r}}{r^3} = 0$ and $\text{div} \frac{\vec{r}}{r} = 3$.

Sol: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}, \quad \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

$$\text{div} \frac{\vec{r}}{r^3} = \frac{\nabla \cdot \vec{r}}{r^3} = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \frac{\vec{r}}{r^3}$$

$$= \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \left[\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r^3} \right]$$

$$= \frac{\partial}{\partial x} \cdot \frac{x}{r^3} + \frac{\partial}{\partial y} \cdot \frac{y}{r^3} + \frac{\partial}{\partial z} \cdot \frac{z}{r^3}$$

Consider $\frac{\partial}{\partial x} \cdot \frac{x}{r^3} = \frac{r^3(1) - x \cdot 3r^2 \cdot \frac{\partial r}{\partial x}}{(r^3)^2} = \frac{r^3 - 3x^2}{r^6}$

$$= \frac{r^3 - 3x^2}{r^6} = \frac{1}{r^3} - \frac{3x^2}{r^6}$$

$$\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) = \frac{1}{r^3} - \frac{3x^2}{r^6} \longrightarrow \textcircled{1}$$

$$\text{Similarly } \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) = \frac{1}{r^3} - \frac{3y^2}{r^6} \longrightarrow \textcircled{2}$$

$$\frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) = \frac{1}{r^3} - \frac{3z^2}{r^6} \longrightarrow \textcircled{3}$$

Add $\textcircled{1}$, $\textcircled{2}$ & $\textcircled{3}$,

$$\frac{\partial}{\partial x} \left(\frac{x}{r^3} \right) + \frac{\partial}{\partial y} \left(\frac{y}{r^3} \right) + \frac{\partial}{\partial z} \left(\frac{z}{r^3} \right) = \frac{3}{r^3} - \frac{3}{r^6} (x^2 + y^2 + z^2)$$

$$= \frac{3}{r^3} - \frac{3}{r^5} \cdot r^2 = \frac{3}{r^3} - \frac{3}{r^3} = 0.$$

$$\operatorname{div} \frac{\vec{r}}{r^3} = 0.$$

Q7: If $\vec{a}$ (Vector) is a Constant Vector, then Prove that $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$.

Sol: let. $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\begin{aligned} \vec{a} \cdot \vec{r} &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) \\ &= a_1x + a_2y + a_3z \end{aligned}$$

$$\begin{aligned} \nabla(\vec{a} \cdot \vec{r}) &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (a_1x + a_2y + a_3z) \\ &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} = \vec{a}. \end{aligned}$$

$$\nabla(\vec{a} \cdot \vec{r}) = \vec{a}.$$

Q8: If $V = (x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}$, then show 'V' is Solenoidal.

$$\begin{aligned} \text{Sol: } \nabla \cdot V &= \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] \cdot [(x+3y)\hat{i} + (y-3z)\hat{j} + (x-2z)\hat{k}] \\ &= \frac{\partial}{\partial x} (x+3y) + \frac{\partial}{\partial y} (y-3z) + \frac{\partial}{\partial z} (x-2z) \\ &= 1 + 1 - 2 = 0. \end{aligned}$$

$$\therefore \nabla \cdot V = 0$$

$\therefore$ 'V' is Solenoidal.

Q9: Find m if $\vec{a} = (x+2y)\mathbf{i} + (my+4z)\mathbf{j} + (5z+6x)\mathbf{k}$ is Solenoidal.

Sol: Since $\vec{a}$ is a solenoidal.

$$\nabla \cdot \vec{a} = 0.$$

$$\left[\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right] \left[(x+2y)\mathbf{i} + (my+4z)\mathbf{j} + (5z+6x)\mathbf{k} \right] = 0$$

$$\Rightarrow \frac{\partial}{\partial x} (x+2y) + \frac{\partial}{\partial y} (my+4z) + \frac{\partial}{\partial z} (5z+6x) = 0$$

$$1+m+5=0$$

$$m+6=0$$

$$m = -6.$$

CURL: If $V(x, y, z)$ is continuously differential vector point function then " $\nabla \times V$ " is called "Curl of V ".

$$\nabla \times V = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

where $V = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$.

Note: A vector point function ' V ' is said to be irrotational if $\text{curl } \vec{V} = 0$.

SOME IMPORTANT PROPERTIES OF CURL:

- If ' V ' is a constant vector, then $\text{curl } V = 0$.
- $\text{curl}(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 \text{curl } V_1 + \alpha_2 \text{curl } V_2$, where $V_1 = V_1(x, y, z)$
 $V_2 = V_2(x, y, z)$
- $\text{curl}(\phi V) = (\text{grad } \phi) \times V + \phi \text{curl } V$
- $\text{curl}(V_1 \times V_2) = V_1(\nabla \cdot V_2) - V_2(\nabla \cdot V_1) + (V_2 \cdot \nabla)V_1 - (V_1 \cdot \nabla)V_2$.

Q1: If V is a vector point function, then s.t $\text{div}(\text{curl } V) = 0$.

Sol: Let $V = V_1 i + V_2 j + V_3 k$

$$\text{curl } V = \nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i \left(\frac{\partial}{\partial y} V_3 - \frac{\partial}{\partial z} V_2 \right) - j \left(\frac{\partial}{\partial x} V_3 - \frac{\partial}{\partial z} V_1 \right) + k \left(\frac{\partial}{\partial x} V_2 - \frac{\partial}{\partial y} V_1 \right)$$

$$= i \left(\frac{\partial}{\partial y} V_3 - \frac{\partial}{\partial z} V_2 \right) + j \left(\frac{\partial}{\partial z} V_1 - \frac{\partial}{\partial x} V_3 \right) + k \left(\frac{\partial}{\partial x} V_2 - \frac{\partial}{\partial y} V_1 \right)$$

$$\text{div}(\text{curl } V) = \nabla \cdot (\nabla \times V)$$

$$= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \left[i \left(\frac{\partial}{\partial y} V_3 - \frac{\partial}{\partial z} V_2 \right) + j \left(\frac{\partial}{\partial z} V_1 - \frac{\partial}{\partial x} V_3 \right) + k \left(\frac{\partial}{\partial x} V_2 - \frac{\partial}{\partial y} V_1 \right) \right]$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial}{\partial y} V_3 - \frac{\partial}{\partial z} V_2 \right) + \frac{\partial}{\partial y} \left(\frac{\partial}{\partial z} V_1 - \frac{\partial}{\partial x} V_3 \right) + \frac{\partial}{\partial z} \left(\frac{\partial}{\partial x} V_2 - \frac{\partial}{\partial y} V_1 \right)$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}$$

$$\text{div}(\text{curl } V) = 0.$$

Q2: If V is a vector point function, then s.t $\text{curl}(\text{curl } V) = \nabla(\nabla \cdot V) - \nabla^2 V$.

Sol: Let $V = V_1 i + V_2 j + V_3 k$

$$\text{curl } V = \nabla \times V = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + j \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) + k \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right)$$

$$\text{curl}(\text{curl} v) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} & \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} & \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \sum i \left\{ \frac{\partial}{\partial y} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \right\}$$

$$= \sum i \left[\frac{\partial^2 v_2}{\partial y \partial x} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2} + \frac{\partial^2 v_3}{\partial z \partial x} \right]$$

$$= \sum i \left[\frac{\partial^2 v_2}{\partial y \partial x} - \frac{\partial^2 v_1}{\partial y^2} - \frac{\partial^2 v_1}{\partial z^2} + \frac{\partial^2 v_1}{\partial x^2} - \frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_3}{\partial z \partial x} \right]$$

$$= \sum i \left[- \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) + \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_2}{\partial y \partial x} + \frac{\partial^2 v_3}{\partial z \partial x} \right) \right]$$

$$= \sum i \left[\frac{\partial}{\partial x} \left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \right) - \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2} \right) \right]$$

$$= \sum i \left\{ \frac{\partial}{\partial x} (\nabla \cdot v) - \nabla^2 v_1 \right\}$$

$$= i \frac{\partial}{\partial x} (\nabla \cdot v) - i \nabla^2 v_1 + j \frac{\partial}{\partial y} (\nabla \cdot v) - j \nabla^2 v_2 + k \frac{\partial}{\partial z} (\nabla \cdot v) - k \nabla^2 v_3$$

$$= \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \nabla \cdot v - \left[i \nabla^2 v_1 + j \nabla^2 v_2 + k \nabla^2 v_3 \right]$$

$$\text{curl}(\text{curl} v) = \nabla(\nabla \cdot v) - \nabla^2 v.$$

Q6: If $\vec{F} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$, & $\vec{F} \cdot \text{curl } \vec{F} = 0$.

Sol: $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+y+1) & 1 & -(x+y) \end{vmatrix}$

$$= \vec{i} \left[\frac{\partial}{\partial y} (x+y) - \frac{\partial}{\partial z} (1) \right] - \vec{j} \left[\frac{\partial}{\partial x} (x+y) - \frac{\partial}{\partial z} (x+y+1) \right] + \vec{k} \left[\frac{\partial}{\partial x} (1) - \frac{\partial}{\partial y} (x+y+1) \right]$$

$$= \vec{i}(-1-0) - \vec{j}(1-0) + \vec{k}(0-1) = -\vec{i} - \vec{j} - \vec{k}$$

$$\begin{aligned} \vec{F} \cdot \text{curl } \vec{F} &= ((x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}) \cdot (-\vec{i} - \vec{j} - \vec{k}) \\ &= -(x+y+1) - 1 - (x+y) \\ &= -x - y - 1 - 1 - x - y = 0 \end{aligned}$$

$$\therefore \vec{F} \cdot \text{curl } \vec{F} = 0$$

Q7: If $\vec{F} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$, then find $\text{div } \vec{F}$, $\text{curl } \vec{F}$ and $\text{curl}[\text{curl } \vec{F}]$.

Sol: Given, $\vec{F} = x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$

$$\begin{aligned} \text{div } \vec{F} &= \nabla \cdot \vec{F} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \vec{F} \\ &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (x^2y\vec{i} - 2xz\vec{j} + 2yz\vec{k}) \\ &= \frac{\partial}{\partial x} (x^2y) - \frac{\partial}{\partial y} (2xz) + \frac{\partial}{\partial z} (2yz) \\ &= 2xy - 0 + 2y \\ &= 2y(x+1) \end{aligned}$$

$$\text{div } \vec{F} = 2xy + 0 + 0y = 2(xy + y)$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= \mathbf{i} \left(\frac{\partial}{\partial y} (2yz) - \frac{\partial}{\partial z} (-2xz) \right) - \mathbf{j} \left(\frac{\partial}{\partial x} (x^2y) - \frac{\partial}{\partial z} (2yz) \right) + \mathbf{k} \left(\frac{\partial}{\partial x} (-2xz) - \frac{\partial}{\partial y} (x^2y) \right)$$

$$= \mathbf{i} (2z + 2x) - \mathbf{j} (x^2 - 2y) + \mathbf{k} (-2z - x^2y)$$

$$\text{curl } \vec{F} = \mathbf{i} (2z + 2x) - \mathbf{j} (x^2 - 2y) + \mathbf{k} (-2z - x^2y)$$

Q3.

Sol:

Q4:

Sol

If 'V' is a vector point function, then $\text{Curl}[\text{grad } \phi] = 0$.

$$\begin{aligned}\text{grad } \phi &= \nabla \phi \\ &= \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] \phi\end{aligned}$$

$$\nabla \phi = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

$$\text{Curl}[\text{grad } \phi] = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial y} \right] - j \left[\frac{\partial}{\partial x} \frac{\partial \phi}{\partial z} - \frac{\partial}{\partial z} \frac{\partial \phi}{\partial x} \right] + k \left[\frac{\partial}{\partial x} \frac{\partial \phi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \phi}{\partial x} \right]$$

$$= \frac{i \partial^2 \phi}{\partial y \partial z} - \frac{i \partial^2 \phi}{\partial z \partial y} + \frac{j \partial^2 \phi}{\partial z \partial x} - \frac{j \partial^2 \phi}{\partial x \partial z} + \frac{k \partial^2 \phi}{\partial x \partial y} - \frac{k \partial^2 \phi}{\partial y \partial x}$$

$$\text{Curl}[\text{grad } \phi] = 0.$$

Find a, b, c if vector $F = (x+2y+az)i + (bx-3y-z)j + (4x+2z+cy)k$ is irrotational.

Given, $\vec{F} = (x+2y+az)i + (bx-3y-z)j + (4x+2z+cy)k$

Since $\vec{F}$ is irrotational, then $\text{Curl } \vec{F} = 0$
 $\nabla \times \vec{F} = 0$.

$$\text{i.e.} \quad \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y+az) & (bx-3y-z) & (4x+2z+cy) \end{vmatrix} = 0$$

$$= i \left[\frac{\partial}{\partial y} (ax + cy + az) - \frac{\partial}{\partial z} (bx - 3y - z) \right] - j \left[\left(\frac{\partial}{\partial x} (ax + cy + az) - \frac{\partial}{\partial z} (x + 2y + az) \right) \right] + k \left[\frac{\partial}{\partial x} (bx - 3y - z) - \frac{\partial}{\partial y} (x + 2y + az) \right]$$

$$\Rightarrow i(c+1) - j(a-4) + k(b-2) = 0$$

$$i(c+1) + j(a-4) + k(b-2) = 0$$

$$\therefore a=4, b=2, c=-1$$

Q5: If $\vec{F} = (\sin y + z)\vec{i} + (x \cos y - z)\vec{j} + (x - y)\vec{k}$ is irrotational.

Sol:

$$\text{Curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\sin y + z) & (x \cos y - z) & (x - y) \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y} (x - y) - \frac{\partial}{\partial z} (x \cos y - z) \right] - j \left[\frac{\partial}{\partial x} (x - y) - \frac{\partial}{\partial z} (\sin y + z) \right] + k \left[\frac{\partial}{\partial x} (x \cos y - z) - \frac{\partial}{\partial y} (\sin y + z) \right]$$

$$= i(-1 - (-1)) - j(1 - 1) + k(\cos y - \cos y)$$

$$= 0$$

$\therefore \vec{F}$ is irrotational.

POTENTIAL FUNCTION

→ Given $\vec{F}$ is a vector point function, if there exists a scalar point function ϕ such that $\vec{F} = \nabla\phi$, then ϕ is said to be scalar point function of $\vec{F}$.

In such case, $\text{curl } \vec{F} = 0$ i.e. $\nabla \times \vec{F} = 0$.

i.e. $\vec{F}$ is an irrotational.

→ If $\vec{F}$ is irrotational, then there exists a scalar potential function ϕ such that $\vec{F} = \nabla\phi$.

$$\text{i.e. } F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k} = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}.$$

Equating like vectors,

$$F_1 = \frac{\partial\phi}{\partial x}, \quad F_2 = \frac{\partial\phi}{\partial y}, \quad F_3 = \frac{\partial\phi}{\partial z}.$$

The scalar potential function ϕ is determined by the above relation by using exact differentials i.e.,

$$d\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz \quad \text{and}$$

Integrate.

Q1: Show that the following Vector $(x^2-yz)i + (y^2-zx)j + (z^2-xy)k$ is irrotational. Find the Scalar potential function.

Sol: Given, $\vec{F} = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2-yz) & (y^2-zx) & (z^2-xy) \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y} (z^2-xy) - \frac{\partial}{\partial z} (y^2-zx) \right] + j \left[\frac{\partial}{\partial z} (x^2-yz) - \frac{\partial}{\partial x} (y^2-zx) \right] + k \left[\frac{\partial}{\partial x} (y^2-zx) - \frac{\partial}{\partial y} (x^2-yz) \right]$$

$$= i [-x - (-x)] + j [-y - (-y)] + k [-z - (-z)]$$

$$= 0.$$

$\therefore \vec{F}$ is irrotational.

If $\vec{F}$ is an irrotational, then $F = \nabla \phi$

$$(x^2-yz)i + (y^2-zx)j + (z^2-xy)k = i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z}$$

Equating like vectors,

$$x^2-yz = \frac{\partial \phi}{\partial x}, \quad y^2-zx = \frac{\partial \phi}{\partial y}, \quad z^2-xy = \frac{\partial \phi}{\partial z}$$

$$\text{W.K.T } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$d\phi = (x^2-yz)dx + (y^2-zx)dy + (z^2-xy)dz$$

Integrating on both sides,

$$d\phi = x^2 dx + y^2 dy + z^2 dz - (yz dx + xz dy + xy dz)$$

$$d\phi = x^2 dx + y^2 dy + z^2 dz$$

$$\int d\phi = \int (x^2 - yz) dx + \int (y^2 - zx) dy + \int (z^2 - xy) dz$$

$$\phi = \frac{x^3}{3} - yzx + \frac{y^3}{3} - zxy + \frac{z^3}{3} - xyz$$

$$\phi = \frac{x^3 + y^3 + z^3}{3} - 3xyz$$

Find the Constants a, b, c of Vector $V = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$. Find ϕ such that $V = \nabla\phi$

Given $\vec{V}$ is irrotational,

$$\text{curl } \vec{V} = 0$$

$$\nabla \times \vec{V} = 0$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x + 2y + az) & (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix} = 0$$

$$= i \left[\frac{\partial}{\partial y} (4x + cy + 2z) - \frac{\partial}{\partial z} (bx - 3y - z) \right] + j \left[\frac{\partial}{\partial z} (x + 2y + az) - \frac{\partial}{\partial x} (4x + cy + 2z) \right] + k \left[\frac{\partial}{\partial x} (bx - 3y - z) - \frac{\partial}{\partial y} (x + 2y + az) \right]$$

$$= i(c + 1) + j(4 - a) + k(b - 2) = 0$$

$$c + 1 = 0$$

$$4 - a = 0$$

$$b - 2 = 0$$

$$c = -1$$

$$a = 4$$

$$b = 2$$

$$V = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k$$

$$V = \nabla \phi$$

$$(x+2y+4z)i + (2x-3y-2)j + (4x-y+2z)k = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \phi$$

Equating like vectors,

$$x+2y+4z = \frac{\partial \phi}{\partial x}, \quad 2x-3y-2 = \frac{\partial \phi}{\partial y}, \quad 4x-y+2z = \frac{\partial \phi}{\partial z}$$

$$\text{w.k.T } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$d\phi = (x+2y+4z)dx + (2x-3y-2)dy + (4x-y+2z)dz$$

Integrating on both sides,

$$\int d\phi = \int (x+2y+4z)dx + \int (2x-3y-2)dy + \int (4x-y+2z)dz$$

$$\phi = \frac{x^2}{2} + 2xy + 4xz + 2xy - \frac{3y^2}{2} - 2y + 4xz - yz + \frac{2z^2}{2}$$

$$= \frac{x^2}{2} - \frac{3y^2}{2} + \frac{2z^2}{2} + 4xy + 8xz - 2yz$$

$$\phi = \frac{x^2 - 3y^2 + 2z^2}{2} + 2(2xy + 4xz) - yz$$

Q3. If $\vec{F} = yzi + zxj + xyk$ is irrotational and find ϕ so that $\vec{F} = \nabla \phi$.

sol:

$$\text{Given, } \vec{F} = yzi + zxj + xyk$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y} (xy) - \frac{\partial}{\partial z} (zx) \right] + j \left[\frac{\partial}{\partial z} (yz) - \frac{\partial}{\partial x} (xy) \right] + k \left[\frac{\partial}{\partial x} (xz) - \frac{\partial}{\partial y} (yz) \right]$$

$$= i(x-x) + j(-y+y) + k(z-z) = 0.$$

$\therefore \vec{F}$ is irrotational.

$$\vec{F} = \nabla \phi$$

$$(yz)i + (zx)j + (xy)k = \left[i \frac{\partial \phi}{\partial x} + j \frac{\partial \phi}{\partial y} + k \frac{\partial \phi}{\partial z} \right]$$

$$\frac{\partial \phi}{\partial x} = yz, \quad \frac{\partial \phi}{\partial y} = zx, \quad \frac{\partial \phi}{\partial z} = xy.$$

$$\text{W.K.T } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$d\phi = yz dx + zx dy + xy dz.$$

Integrating on both sides, $= \int d(xyz)$

$$\int d\phi = \int yz dx + \int zx dy + \int xy dz =$$

$$\phi = xyz + xyz + xyz$$

$$\phi = \underline{xyz}.$$

⁵⁷
Q4: $\vec{F} = (y^2 \cos x + z^3)i + (2y \sin x - 4)j + (3xz^2 + 2)k$ is Conservative and find its scalar potential.

Sol: Given,

$$\vec{F} = (y^2 \cos x + z^3)i + (2y \sin x - 4)j + (3xz^2 + 2)k$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y^2 \cos x + z^3) & (2y \sin x - 4) & (3xz^2 + 2) \end{vmatrix}$$

$$= i \left[\frac{\partial}{\partial y} (3xz^2 + 2) - \frac{\partial}{\partial z} (2y \sin x - 4) \right] + j \left[\frac{\partial}{\partial z} (y^2 \cos x + z^3) - \frac{\partial}{\partial x} (3xz^2 + 2) \right] \\ + k \left[\frac{\partial}{\partial x} (2y \sin x - 4) - \frac{\partial}{\partial y} (y^2 \cos x + z^3) \right]$$

$$= 0i - j(3z^2 - 3z^2) + k(2y \cos x - 2y \cos x) = 0$$

$\therefore \vec{F}$ is conservative.

$$\vec{F} = \nabla \phi$$

$$(y^2 \cos x + z^3)i + (2y \sin x - 4)j + (3xz^2 + 2)k = \left[\frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k \right]$$

$$\frac{\partial \phi}{\partial x} = y^2 \cos x + z^3, \quad \frac{\partial \phi}{\partial y} = 2y \sin x - 4, \quad \frac{\partial \phi}{\partial z} = 3xz^2 + 2.$$

$$\text{W.K.T, } d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz$$

$$d\phi = (y^2 \cos x + z^3) dx + (2y \sin x - 4) dy + (3xz^2 + 2) dz$$

Integrating on both sides,

$$\int d\phi = \int (y^2 \cos x + z^3) dx + \int (2y \sin x - 4) dy + \int (3xz^2 + 2) dz$$

$$= y^2 \sin x + z^3 x + 2 \sin x y^2 - 4y + 3x \cdot \frac{z^3}{3} + 2z$$

$$= 2 \sin x y^2 + x z^3 - 4y + 2z.$$

$$\phi = 2(y^2 \sin x + z^3 x - 2y + z).$$

(or)

$$\int d\phi = \int y^2 \cos x + z^3 dx + \int 2y \sin x dy - 4 \int dy + 3x \int z^2 dz + 2 \int dz$$

$$= \int (y^2 \cos x dx + 2y \sin x dy) + \int (z^3 dx + 3xz^2) - 4 \int dy + 2 \int dz$$

$$= \int d(y^2 \sin x) + \int d(x z^3) - 4 \int dy + 2 \int dz$$

$$\phi = y^2 \sin x + x z^3 - 4y + 2z.$$

LINE, SURFACE AND VOLUME INTEGRAL

LINE INTEGRAL: Any integral which is to be evaluated along a curve is called a line integral. Let $F(x, y, z) = F_1i + F_2j + F_3k$ be a continuous vector point function at every point of curve in space. It is denoted by $\int_C \vec{F} \cdot d\vec{r}$

Note: $\vec{F} = F_1i + F_2j + F_3k$, $\vec{r} = xi + yj + zk$.

① $\rightarrow d\vec{r} = dx i + dy j + dz k$.

$$\int_C \vec{F} \cdot d\vec{r} = \int F_1 dx + F_2 dy + F_3 dz.$$

② $\rightarrow$ If a curve is in parametric form, $x = x(t)$, $y = y(t)$, $z = z(t)$, let the end points of curve $t = t_1$ at A, $t = t_2$ at B.

then $\int_C \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt$.

③ $\rightarrow$ If "C" is closed curve, then Integral is defined by

$$\oint_C \vec{F} \cdot d\vec{r}$$

④ $\rightarrow$ If "C" is simple closed curve, then the Tangent line ($\oint_C \vec{F}$) integral of $\vec{F}$ vector around 'C' is called Circulation of C vector.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint F_1 dx + F_2 dy + F_3 dz$$

If $\oint_C \vec{F} \cdot d\vec{r} = 0$, then $\vec{F}$ is irrotational.

⑤ $\rightarrow$ Work done by the Force:

If F represents the force acting on a particle moving along the arc AB, the work done during the displacement

Q1 δr is $\delta r = \vec{F} \cdot d\vec{r}$.

The total work done by the force during the displacement from A to B

$$\int_A^B \vec{F} \cdot d\vec{r} = \int_A^B \vec{F} \cdot d\vec{r}$$

⑥ → The line integral is independent of path if $\vec{F}$ is Conservative field or irrotational, then $\vec{F} = \nabla\phi$.

Q1: If $\vec{F} = [5xy - 6x^2]i + [2y - 4x]j$. Evaluate $\int \vec{F} \cdot d\vec{r}$ along the Curve 'C' in xy-plane, $y = x^3$ from point (1,1) to (2,8).

Sol: Given, $\vec{r} = xi + yj + zk$

$$d\vec{r} = dx i + dy j + dz k$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \{(5xy - 6x^2)i + (2y - 4x)j\} \cdot \{dx i + dy j + dz k\}$$

$$y = x^3$$

$$dy = 3x^2 dx$$

$$= \int_C \{(5x \cdot x^3 - 6x^2)i + (2x^3 - 4x)j\} \cdot \{dx i + dy j + dz k\}$$

$$= \int_C (5x^4 - 6x^2) dx + (2x^3 - 4x) dy$$

$$= \int_C \{5x^4 - 6x^2 + (2x^3 - 4x)3x^2\} dx$$

$$= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3) dx$$

$$= \left[\left(\frac{5x^5}{5} \right) - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4} \right]_1^2$$

$$= (32-1) - [16-2] + [64-1] - 3[16-1]$$

$$= 31 - 14 + 63 - 45 = 35.$$

$$\int_C \vec{F} \cdot d\vec{r} = 35.$$

Q2: Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2\vec{i} + y^3\vec{j}$ and curve 'C' is arc of parabola $y = x^2$ in xy-plane from (0,0) to (1,1).

Sol: Given, $\vec{F} = x^2\vec{i} + y^3\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int (x^2\vec{i} + y^3\vec{j}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int x^2 dx + y^3 dy$$

$$y = x^2$$

$$dy = 2x dx$$

$$\int_C \vec{F} \cdot d\vec{r} = \int x^2 dx + (x^2)^3 2x dx = \int (x^2 + 2x^7) dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{2x^8}{8}\right)_0^1 = \frac{1}{3} + \frac{2}{8} = \frac{8+6}{24} = \frac{7}{12}.$$

Q3: If $\vec{F} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$, where 'C' is a straight line joining the points (0,0,0) to (1,1,1).

Sol: let A(0,0,0) B(1,1,1)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t.$$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$$

$$x = y = z = t$$

$$dx = dy = dz = dt$$

$$\int \vec{F} \cdot d\vec{r} = \int ((3x^2 + 6y)\vec{i} - (4yz)\vec{j} + (20xz^3)\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int (3x^2 + 6y)dx - 4yzdy + 20xz^3dz$$

$$= \int (3t^2 + 6t)dt - 14t^2dt + 20t^3dt$$

$$= \int (-11t^2 + 6t + 20t^3)dt$$

$$= \left[\frac{20t^4}{4} - \frac{11t^3}{3} + \frac{6t^2}{2} \right]_0^1$$

$$= 5(1) - \frac{11}{3}(1) + 3(1) = 13/3$$

Q4: Find the work done by the force field $\vec{F} = (x^2 - xy)\vec{i} + (y^2 - 2xy)\vec{j}$ in moving the particle from point $(1, 1)$ to $(2, 4)$ along the curve $y = x^2$.

Sol: Work done $\delta r = \vec{F} \cdot d\vec{r}$

$$= [(x^2 - xy)\vec{i} + (y^2 - 2xy)\vec{j}] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}]$$

$$= (x^2 - xy)dx + (y^2 - 2xy)dy$$

$$\rightarrow y = x^2$$

$$dy = 2x dx$$

$$= (x^2 - x \cdot x^2) dx + (x^3 - 2x \cdot x^2) 2x dx$$

$$= (x^2 - x^3 + 2x^5 - 4x^4) dx = \int_1^2 (x^2 - x^3 + 2x^5 - 4x^4) dx$$

on Integrating,

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{2x^6}{6} - \frac{4x^5}{5} \right]_1^2$$

$$= \left[\frac{8}{3} - \frac{16}{4} + \frac{2 \times 64}{43} - \frac{4 \times 32}{5} \right] - \left[\frac{1}{3} - \frac{1}{4} + \frac{2}{6} - \frac{4}{5} \right]$$

$$= \frac{8}{3} - 4 + \frac{64}{3} - \frac{128}{5} - \frac{1}{3} + \frac{1}{4} - \frac{1}{3} + \frac{4}{5}$$

$$= \frac{6}{3} + \frac{64}{3} - \frac{128}{5} - \frac{15}{4} + \frac{4}{5} = \frac{180 + 1280 - 1536}{60}$$

$$= -\frac{313}{60}$$

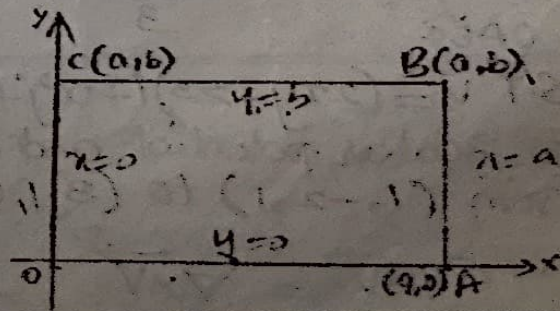
Q5: Evaluate $\int \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\vec{j}$, C is the rectangle in xy -plane and bounded by $y=0$, $x=0$, $x=a$, $y=b$.

Sol: $\int \vec{F} \cdot d\vec{r} = \int \{ (x^2 + y^2)\vec{i} - (2xy)\vec{j} \} \{ dx\vec{i} + dy\vec{j} \}$
 $= \int (x^2 + y^2) dx - 2xy dy \quad \text{--- (1)}$

Along $\vec{OA}$: $y=0$

$dy=0$, $x: 0 \rightarrow a$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_{OA} x^2 dx$$



$$= \int_0^a x^2 dx = \left(\frac{x^3}{3} \right)_0^a = \frac{a^3}{3} \quad \text{--- } (2)$$

Along $\overrightarrow{AB}$: $x=a$ $y:0 \rightarrow b$
 $dx=0$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_{AB} -2axy dy = -2a \int_0^b y dy = -2a \left[\frac{y^2}{2} \right]_0^b$$

$$= \frac{-2ab^2}{2} = -ab^2.$$

Along $\overrightarrow{BC}$: $y=b$ $x:a \rightarrow 0$
 $dy=0$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_{BC} (x^2 + y^2) dx = \int_a^0 (x^2 + b^2) dx = \left(\frac{x^3}{3} + b^2x \right)_a^0$$

$$= -\frac{a^3}{3} - ab^2.$$

Along $\overrightarrow{CD}$: $x=0$ $y:b \rightarrow 0$
 $dx=0$

$$\int_{CD} \vec{F} \cdot d\vec{r} = \int_{CD} -2axy dy = 0.$$

$$\therefore \int_{OABC} \vec{F} \cdot d\vec{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2.$$

Q6: S.T $\vec{V} = (2xy + z^3)\vec{i} + x^2\vec{j} + 3xz^2\vec{k}$ is conservative field. Find its scalar potential and also the work done in moving from $(1, -2, 1)$ to $(3, 1, 4)$.

Sol: $\text{Curl } \vec{V} = \nabla \times \vec{V}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy+z^3) & x^2 & 3xz^2 \end{vmatrix}$$

$$= i(0-0) + j(3z^2-3z^2) + k(2x-2x) \\ = 0.$$

$$\text{Curl } \vec{V} = 0.$$

$\therefore \vec{V}$ is a Conservative field.

If $\vec{V}$ is Conservative field then, $\vec{V} = \nabla \phi$.

$$(2xy+z^3)i + x^2j + 3xz^2k = i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}.$$

Equating like Vectors,

$$\frac{\partial \phi}{\partial x} = 2xy+z^3, \quad \frac{\partial \phi}{\partial y} = x^2, \quad \frac{\partial \phi}{\partial z} = 3xz^2$$

$$\text{W.K.T, } d\phi = \frac{\partial \phi}{\partial x} \cdot dx + \frac{\partial \phi}{\partial y} \cdot dy + \frac{\partial \phi}{\partial z} \cdot dz$$

$$= (2xy+z^3)dx + x^2dy + 3xz^2dz$$

$$d\phi = 2xydx + z^3dx + x^2dy + 3xz^2dz$$

Integrating on both sides,

$$\int d\phi = \int d(x^2y) + \int d(z^3x)$$

$$\phi = x^2y + z^3x$$

$$\begin{aligned} \text{The work done by force} &= \int_A^B d\phi = [\phi]_A^B = [x^2y + z^3x]_A^B \\ &= (x^2y + z^3x)_{(3,1,4)}^{(1,-2,1)} = (9 + (64 \times 3)) - (-2 + 1) \\ &= 201 + 1 = 202. \end{aligned}$$

SURFACE INTEGRALS

An Integral which is to be evaluated over a surface is called a surface integral.

Suppose 'S' is the surface area

Suppose $f(x, y, z)$ is single value function over the surface 'S'.

Note: To evaluate any surface integral, it is convenient to evaluate double integral of its projection in xy-plane, yz-plane or xz-plane.

$$\rightarrow \iint_S \vec{F} \cdot \vec{n} \, ds = \iint_S \vec{F} \cdot \vec{ds} = \iint_R \vec{F} \cdot \vec{n} \frac{dx \, dy}{|\vec{n} \cdot \vec{k}|}$$

where 'R' is the projection in xy-plane.

$$\rightarrow \iint_S \vec{F} \cdot \vec{n} \, ds = \iint_S \vec{F} \cdot \vec{ds} = \iint_R \vec{F} \cdot \vec{n} \frac{dx \, dz}{|\vec{n} \cdot \vec{j}|}$$

where 'R' is the projection in xz-plane.

$$\rightarrow \iint_S \vec{F} \cdot \vec{n} \, ds = \iint_S \vec{F} \cdot \vec{ds} = \iint_R \vec{F} \cdot \vec{n} \frac{dy \, dz}{|\vec{n} \cdot \vec{i}|}$$

where 'R' is the projection in yz-plane.

Q1: Evaluate $\iint_S \mathbf{A} \cdot \mathbf{n} \, ds$, where $\mathbf{A} = 18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$ and where S is the part of plane $2x + 3y + 6z = 12$, which is the first octant.

Sol: Let $\phi = 2x + 3y + 6z - 12 = 0$

$$\nabla\phi = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) (2x + 3y + 6z - 12)$$

$$= 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$$

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{4 + 9 + 36}} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7}$$

$$\mathbf{n} \cdot \mathbf{k} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7} \cdot \mathbf{k} = \frac{6}{7}$$

$$\mathbf{A} \cdot \mathbf{n} = (18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}) \cdot \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7}$$

$$\Rightarrow \frac{36z}{7} - \frac{36}{7} + \frac{18y}{7}$$

$$= \frac{1}{7} \left[36 \left(\frac{12 - 2x - 3y}{6} \right) - 36 + 18y \right]$$

$$= \frac{(72 - 12x - 18y - 36 + 18y)}{7}$$

$$= \frac{-12x + 36}{7} = \frac{6(6 - 2x)}{7}$$

$$\iint_S \mathbf{A} \cdot \mathbf{n} \, ds = \iint_S \mathbf{A} \cdot \hat{\mathbf{n}} \frac{dx \, dy}{|\mathbf{n} \cdot \mathbf{k}|} = \iint_R \frac{1}{7} (36 - 12x) \frac{dx \, dy}{(6/7)}$$

$$\iint (6 - 2x) \, dx \, dy$$

$$\left| \begin{array}{l} 2x + 3y + 6z = 12 \\ z = \frac{12 - 2x - 3y}{6} \end{array} \right.$$

$$\int_0^6 \int_{\frac{12-2x}{3}}^{\frac{12-2x}{3}} (6-2x) dx dy$$

$$= \int_0^6 (6-2x) (y)_0^{\frac{12-2x}{3}} dx$$

$$2x+3y=12$$

$$y=0$$

$$x=6$$

$$y=\frac{12-2x}{3}$$

$$= \int_0^6 \frac{(6-2x)(12-2x)}{3} dx = \frac{1}{3} \int_0^6 (4x^2 - 12x - 24 + 4x^2) dx$$

$$= \frac{1}{3} \int_0^6 (4x^2 - 36x + 72) dx = \frac{1}{3} \left[\frac{4x^3}{3} - \frac{36x^2}{2} + 72x \right]_0^6$$

$$= \frac{1}{3} \left[\frac{4 \times 6^3}{3} - \frac{36 \times 36}{2} + 72 \times 6 \right] = 24.$$

$$\iint_S A \cdot nds = 24.$$

Q2: Evaluate $\iint_S \vec{F} \cdot nds$ where $\vec{F} = x\vec{i} + y\vec{j} + 3y^2z\vec{k}$ and 'S' is the surface of cylinder $x^2 + y^2 = 16$ in the first octant degree, $z=0$ and $z=6$.

Sol: Given, $\vec{F} = x\vec{i} + y\vec{j} + 3y^2z\vec{k}$

$$n = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\nabla \phi = \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] (x^2 + y^2 = 16)$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla \phi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2 \times 4 = 8$$

$$\therefore n = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(x\vec{i} + y\vec{j})}{8} = \frac{x\vec{i} + y\vec{j}}{4}$$

$$n \cdot \vec{i} = \frac{\nabla \phi}{|\nabla \phi|} \cdot \vec{i} = \frac{x\vec{i} + y\vec{j}}{4} \cdot \vec{i} = \frac{x}{4}$$

$$\vec{F} \cdot \vec{n} = [zi + xj + 3y^2k] \cdot \left[\frac{zi + yj}{4} \right] = \frac{zx + xy}{4} = \frac{x}{4} (z + y)$$

$$\iint \vec{F} \cdot \vec{n} \, ds = \iint \frac{\vec{F} \cdot \vec{n} \, dy \, dz}{|\vec{n}|} = \iint \frac{x}{4} (z + y) \frac{dy \, dz}{(x/4)}$$

$$\int_{z=0}^4 \int_{y=0}^6 (z + y) \, dy \, dz$$

$$= \int_{z=0}^4 \left(\frac{z^2}{2} + yz \right) \Big|_0^6 \, dz = \int_0^4 (18 + 6y) \, dy = \left(18y + \frac{6y^2}{2} \right) \Big|_0^4$$

$$= 72 + 48 = 120.$$

$$\iint \vec{F} \cdot \vec{n} \, ds = 120.$$

$$\begin{aligned} x^2 + y^2 &= 16 \\ y^2 &= 16, y = 4 \end{aligned}$$

$$\frac{x^2 + y^2}{4}$$

Q3: If $\vec{F} = 2yi - zj + x^2k$ and 'S' is the surface of parabolic cylinder and $y^2 = 8x$ in the first octant bounded by the planes $y=4$, $z=6$. Evaluate $\int_S \vec{F} \cdot \vec{n} \, ds$.

Sol: Given, $\vec{F} = 2yi - zj + x^2k$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\nabla \phi = \left[i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right] [y^2 - 8x] = -8i + 2yj$$

$$|\nabla \phi| = \sqrt{64 + 4y^2} = 2\sqrt{16 + y^2}$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{-8i + 2yj}{2\sqrt{16 + y^2}} = \frac{-4i + yj}{\sqrt{16 + y^2}}$$

$$\vec{n} \cdot \vec{i} = \frac{-4i + yj}{\sqrt{16 + y^2}} \cdot \vec{i} = \frac{-4}{\sqrt{16 + y^2}}$$

$$\begin{aligned}
 \vec{F} \cdot \vec{n} &= (2y^4 \hat{i} - z \hat{j}) \cdot \left(\frac{-4\hat{i} + y\hat{j}}{\sqrt{16+y^2}} \right) = \frac{1}{\sqrt{16+y^2}} (-8y - zy) \\
 \int_S \vec{F} \cdot \vec{n} \, ds &= \iint \frac{\vec{F} \cdot \vec{n}}{|\vec{n}|} \, dy \, dz = \iint \frac{1}{\sqrt{16+y^2}} (-8y - zy) \, dy \, dz \\
 &= \iint \frac{-1}{4} (-8y - zy) \, dy \, dz \\
 &= \frac{1}{4} \int_{y=0}^4 \int_{z=0}^6 (8y + zy) \, dy \, dz = \frac{1}{4} \int_{y=0}^4 \left(8yz + \frac{z^2 y}{2} \right) \Big|_0^6 \, dy \\
 &= \frac{1}{4} \int_{y=0}^4 (48y + \frac{36y}{2}) \, dy = \frac{1}{4} \int_{y=0}^4 (48y + 18y) \, dy \\
 &= \left(\frac{66y^2}{4 \times 2} \right) \Big|_0^4 = \frac{66 \times 16}{4 \times 2} = 132. \\
 \therefore \int \vec{F} \cdot \vec{n} \, ds &= 132.
 \end{aligned}$$

Q4: Evaluate $\iint_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and 'S' is the surface of plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$.

Sol: Given, $\vec{F} = y\hat{i} + 2x\hat{j} - z\hat{k}$
 $2x + y - 6 = 0 \Rightarrow y = 6 - 2x$

$$\nabla \phi = \left[\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right] (2x + y - 6) = 2\hat{i} + \hat{j}$$

$$|\nabla \phi| = \sqrt{4+1} = \sqrt{5}$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}}, \quad \text{N.S.} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}} \cdot \hat{j} = \frac{1}{\sqrt{5}}$$

$$\vec{F} \cdot \vec{n} = (yi + 2xj - zk) \left(\frac{2i + j}{\sqrt{5}} \right) = \frac{2y + 2x}{\sqrt{5}}$$

$$\iint_S \vec{F} \cdot \vec{n} \, ds = \iint_S \frac{\vec{F} \cdot \vec{n}}{|\vec{n} \cdot \vec{j}|} \, dx \, dz = \iint_S \frac{2(x+y)}{\sqrt{5}} \cdot \frac{1}{(\frac{1}{\sqrt{5}})} \, dx \, dz$$

$$= 2 \iint_S (x+y) \, dx \, dz = 2 \int_{x=0}^{x=3} \int_{z=0}^{z=4} (x+6-2x) \, dx \, dz$$

$$= 2 \int_0^3 \int_0^4 (6-x) \, dx \, dz = 2 \int_0^3 (6z - xz)_0^4 \, dz = 2 \int_0^3 (24 - 4x) \, dz$$

$$= 2 \left[24x - \frac{4x^2}{2} \right]_0^3 = 2 [72 - 2(9)] = 2(54) = 108.$$

$$\iint_S \vec{F} \cdot \vec{n} \, ds = 108.$$

Q5: Evaluate $\iint_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = yzi + zxj + xyk$ and 'S' is a part of surface of sphere $x^2 + y^2 + z^2 = 1$ which lies in the first octant.

Sol: $\vec{F} = yzi + zxj + xyk$, $\phi = x^2 + y^2 + z^2 = 1$

$$\iint_S \vec{F} \cdot \vec{n} \, ds = \iint_R \frac{\vec{F} \cdot \vec{n}}{|\vec{n} \cdot \vec{k}|} \, dx \, dy$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\nabla \phi = \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2 - 1)$$

$$= 2xi + 2yj + 2zk.$$

$$|\nabla \phi| = \sqrt{4x^2 + 4y^2 + 4z^2} = 2\sqrt{x^2 + y^2 + z^2} = 2\sqrt{1} = 2.$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2(x\vec{i} + y\vec{j} + z\vec{k})}{2} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{n} \cdot \vec{k} = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot \vec{k} = z, \quad \vec{F} \cdot \vec{n} = [yz\vec{i} + xz\vec{j} + xy\vec{k}] \cdot [x\vec{i} + y\vec{j} + z\vec{k}]$$

$$= xyz + xyz + xyz = 3xyz$$

$$\vec{F} \cdot \vec{n} = 3xyz$$

$$\iint_S \vec{F} \cdot \vec{n} \, ds = \iint_S \frac{3xyz}{2} \, dxdydz = 3 \iint_S xy \, dxdydz$$

$$= 3 \iint_S xy \, dxdydz = 3 \int_0^1 \int_0^{\sqrt{1-x^2}} xy \, dxdydz$$

$$= 3 \int_0^1 \left(\frac{xy^2}{2} \right)_0^{\sqrt{1-x^2}} dx$$

$$= 3 \int_0^1 \frac{x(1-x^2)}{2} dx = \frac{3}{2} \int_0^1 (x - x^3) dx$$

$$= \frac{3}{2} \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{2} \left[\frac{1}{2} - \frac{1}{4} \right] = \frac{3}{2} \cdot \frac{2-1}{4 \times 2} = \frac{3}{8}$$

$$\iint_S \vec{F} \cdot \vec{n} \, ds = \frac{3}{8}$$

Q6: Evaluate $\iint_S \vec{F} \cdot \vec{n} \, ds$ where $\vec{F} = z^2\vec{i} + x^2\vec{j} - y^2\vec{k}$ and S be the surface of plane $x^2 + y^2 = 16$ in the first Octant cut off by the plane $z=0, z=5$.

Sol: Given, $\vec{F} = z^2\vec{i} + x^2\vec{j} - y^2\vec{k}$

$$\phi = x^2 + y^2 = 16$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|}, \quad \nabla \phi = \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] [x^2 + y^2 = 16]$$

$$\eta = 2xi + 2yj$$

$$|\nabla\phi| = \sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2} = 2\sqrt{16} = 2(4) = 8.$$

$$\eta = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi + yj)}{8} = \frac{xi + yj}{4}.$$

$$\eta \cdot j = \left(\frac{xi + yj}{4}\right) \cdot j = \frac{y}{4}.$$

$$F \cdot \eta = (z^2i + x^2j - y^2k) \cdot \left(\frac{xi + yj}{4}\right) = \frac{z^2x + x^2y}{4}$$

$$\iint_S F \cdot \eta \, ds = \iint_R \frac{z^2x + x^2y}{4} \cdot \frac{dx \, dz}{(y/4)} = \iint_R \left(\frac{z^2x}{\sqrt{16-x^2}} + x^2 \right) dx \, dz$$

$$= \int_0^4 \int_0^5 \left(\frac{z^2x}{\sqrt{16-x^2}} + x^2 \right) dx \, dz$$

$$\begin{aligned} x^2 + y^2 &= 16 \\ y &= \sqrt{16-x^2} \\ y &= 4 \end{aligned}$$

$$= \int_0^4 \left[\frac{x}{\sqrt{16-x^2}} \cdot \frac{z^3}{3} + x^2 z \right]_0^5 dz = \int_0^4 \left[\frac{125}{3} \cdot \frac{x}{\sqrt{16-x^2}} + 5x^2 \right] dx$$

$$= \frac{125}{3} \int_0^4 \frac{x}{\sqrt{16-x^2}} dx + 5 \int_0^4 x^2 dx$$

$$\text{let } 16-x^2 = t$$

$$-2x \, dx = dt$$

$$x \, dx = -dt/2$$

Limits:

$$x \quad 0 \quad 4$$

$$t \quad 16 \quad 0$$

$$= \frac{-1 \times 125}{2 \times 3} \int_{16}^0 \frac{1}{t^{1/2}} dt + \left(\frac{5x^3}{3} \right)_0^4 = \frac{-125}{6} \left[\frac{t^{1/2}}{1/2} \right]_{16}^0 + \frac{5x^3}{3}$$

$$= \frac{-125}{6} \times 2(-4) + \frac{320}{3} = \frac{500}{3} + \frac{320}{3}$$

$$= \frac{820}{3}$$

VOLUME INTEGRAL:

Any integral which is to be evaluated, over a volume is called volume integral.

If $\vec{F}(x, y, z)$ is vector point function, then the volume integral is given by $\iiint_V \vec{F} \cdot d\vec{v}$

Q1: Evaluate $\iiint_V \nabla \cdot \vec{F} dv$ where $\vec{F} = (2x^2 + 4z)\vec{i} - 2xy\vec{j} - 8x^2\vec{k}$ vector and 'V' is bounded by the plane $x=0, y=0, z=0$ and $x+y+z=1$.

Sol: Given, $\vec{F} = (2x^2 + 4z)\vec{i} - 2xy\vec{j} - 8x^2\vec{k}$

$$\phi = x + y + z = 1$$

$$\nabla \cdot \vec{F} = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot ((2x^2 + 4z)\vec{i} - 2xy\vec{j} - 8x^2\vec{k})$$

$$= 4x - 2x - 0 = 2x$$

$$\iiint_V \nabla \cdot \vec{F} dv = \int_{x=0}^1 \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} [2x] dz dy dx$$

$$x + y + z = 1$$

$$z = 1 - x - y$$

$$z = 0$$

$$1 - x - y = 0$$

$$y = 0$$

$$x = 1$$

$$= 2 \int_0^1 \int_0^{1-x} x [z]_0^{1-x-y} dx dy = 2 \int_0^1 \int_0^{1-x} x(1-x-y) dx dy$$

$$= 2 \int_0^1 \int_0^{1-x} (x - x^2 - xy) dx dy = 2 \int_0^1 \left[xy - x^2 y - \frac{xy^2}{2} \right]_0^{1-x} dy$$

$$= 2 \int_0^1 \left[x(1-x) - x^2(1-x) - \frac{x(1-x)^2}{2} \right] dx$$

$$= 2 \int_0^1 \left(x - x^2 - x^2 + x^3 - x \left(\frac{1+x^2-2x}{2} \right) \right) dx$$

$$= 2 \int_0^1 \left(x - 2x^2 + x^3 - \frac{x}{2} - \frac{x^3}{2} + \frac{2x^2}{2} \right) dx$$

$$= 2 \int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2} \right) dx = 2 \left[\frac{1}{2} \cdot \frac{x^2}{2} - \frac{x^3}{3} + \frac{1}{2} \cdot \frac{x^4}{4} \right]_0^1$$

$$= 2 \left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8} \right] = 2 \left[\frac{6-8+3}{24} \right] = \frac{1}{12}$$

Q2: Evaluate $\iiint_V \phi dv$ where $\phi = 45x^2y$ and 'V' is a closed region bounded by a plane $4x+3y+z=8$ and $x=y=z=0$

Sol:

Given,

$$\phi = 4x + 3yz - 8$$

$$\iiint_V \phi dv = \iiint_V 45x^2y dx dy dz$$

$$4x+3y+z=8$$

$$z=8-4x-3y$$

$$z=0, y=\frac{8-4x}{3}$$

$$y=0, x=2$$

$$\int_{x=0}^2 \int_{y=0}^{\frac{8-4x}{3}} \int_{z=0}^{8-4x-3y} 45x^2y dz dy dx$$

$$\int_{x=0}^2 \int_{y=0}^{\frac{8-4x}{3}} \left[45x^2yz \right]_0^{8-4x-3y} dy dx = 45 \int_{x=0}^2 \int_{y=0}^{\frac{8-4x}{3}} x^2y (8-4x-3y) dy dx$$

$$= 45 \int_0^2 \int_0^{\frac{8-4x}{3}} (8x^2y - 4x^3y - 3x^2y^2) dy dx$$

$$= 45 \int_0^2 \left(\frac{8x^4 y^2}{2} - \frac{4x^3 y^2}{2} - \frac{3x^2 y^3}{3} \right) \frac{8-4x}{3} dx$$

$$= 45 \int_0^2 (4x^3 y^2 - 2x^3 y^2 - x^2 y^3) \frac{8-4x}{3} dx$$

$$= 45 \int_0^2 \left(4x^3 \frac{(8-4x)^2}{9} - 2x^3 \frac{(8-4x)^2}{9} - x^2 \frac{(8-4x)^3}{27} \right) dx$$

$$= \frac{45 \times 2}{9} \int_0^2 \left[x^2 \left(4(8-4x)^2 - 2x(8-4x)^2 - \frac{(8-4x)^3}{3} \right) \right] dx$$

$$= 5 \int_0^2 x^2 \left[4(64 + 16x^2 - 64x) - 2x(64 + 16x^2 - 64x) - \frac{(-64x^3 + 384x^2 - 768x + 512)}{3} \right] dx$$

$$= 5 \int_0^2 x^2 \left[256 + 64x^2 - 256x - 128x - 32x^3 + 128x^3 + 64x^3 - 384x^2 + 768x - 512 \right] dx$$

$$= \frac{5}{3} \int_0^2 x^2 \left[768 + 192x^2 - 128x - 384x - 96x^3 + 384x^3 + 64x^3 - 384x^2 + 768x - 512 \right] dx$$

$$= \frac{5}{3} \int_0^2 x^2 \left[352x^3 - 192x^2 - 384x + 256 \right] dx$$

$$= \frac{5 \times 32}{3} \int_0^2 (11x^5 - 6x^4 - 12x^3 + 8x^2) dx$$

$$= \frac{5 \times 32}{3} \left[\frac{11x^6}{6} - \frac{6x^5}{5} - \frac{12x^4}{4} + \frac{8x^3}{3} \right]_0^2 = \frac{5 \times 32}{3} \left[\frac{11 \times 64}{6} - \frac{6 \times 32}{5} - \frac{12 \times 16}{4} + \frac{8 \times 8}{3} \right]$$

$$= \frac{160}{3} \left[\frac{704}{6} - \frac{192}{5} - 48 + \frac{64}{3} \right] = \frac{160}{3} \times \frac{3520 - 1152 - 1440 + 640}{30}$$

Q3. If $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$, where 'V' is bounded by plane $x=0, y=0, z=0$ and $2x+2y+z=4$. $\iiint_V \nabla \cdot \vec{F} dv$

Sol:

Given, $\vec{F} = (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$

$$\phi = 2x + 2y + z - 4$$

$$\begin{aligned}\nabla \cdot \vec{F} &= \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] (2x^2 - 3z)\vec{i} - 2xy\vec{j} - 4x\vec{k} \\ &= \frac{\partial}{\partial x} (2x^2 - 3z) - \frac{\partial}{\partial y} (2xy) + \frac{\partial}{\partial z} (-4x) \\ &= 4x - 2x + 0 = 2x.\end{aligned}$$

$$\iiint_V \nabla \cdot \vec{F} dv = \int_{x=0}^2 \int_{y=0}^{2-x} \int_{z=0}^{4-2x-2y} 2x dx dy dz$$

$2x + 2y + z = 4$
 $z = 4 - 2x - 2y$
 $z=0,$
 $y = 2 - x$
 $y=0$
 $x=2.$

$$= 2 \int_0^2 \int_0^{2-x} x(z)_0^{4-2x-2y} dx dy = 2 \int_0^2 \int_0^{2-x} x(4-2x-2y) dx dy$$

$$= 2 \int_0^2 \int_0^{2-x} (4x - 2x^2 - 2xy) dx dy = 2 \int_0^2 \left[4xy - 2x^2y - \frac{2xy^2}{2} \right]_0^{2-x} dy$$

$$= 2 \int_0^2 \left(4x(2-x) - 2x^2(2-x) - x(2-x)^2 \right) dx$$

$$= 2 \int_0^2 (8x - 4x^2 - 4x^2 + 2x^3 - 4x - x^3 + 4x^2) dx$$

$$\begin{aligned}
 &= 2 \int_0^2 (x^3 - 4x^2 + 4x) dx = 2 \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2} \right)_0^2 \\
 &= 2 \left(\frac{16}{4} - \frac{4 \cdot 8}{3} + 2(4) \right) = 2 \left(4 - \frac{32}{3} + 8 \right) \\
 &= 2 \left(12 - \frac{32}{3} \right) = 2 \times \frac{4}{3} = \frac{8}{3}.
 \end{aligned}$$

Q4: Evaluate $\iiint \nabla \cdot \vec{F} dv$ where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and V is the volume enclosed by cube $0 \leq x, y, z \leq 1$.

Sol.

Given, $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$$\begin{aligned}
 \nabla \cdot \vec{F} &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (x^2\vec{i} + y^2\vec{j} + z^2\vec{k}) \\
 &= 2x + 2y + 2z = 2(x + y + z).
 \end{aligned}$$

$$\iiint \nabla \cdot \vec{F} dv = \iiint 2(x + y + z) dx dy dz$$

$$= 2 \int_0^1 \int_0^1 \int_0^1 (x + y + z) dx dy dz = 2 \int_0^1 \int_0^1 \left(xz + yz + \frac{z^2}{2} \right) dy dz$$

$$= 2 \int_0^1 \int_0^1 \left(x + y + \frac{1}{2} \right) dx dy = 2 \int_0^1 \left(xy + \frac{y^2}{2} + \frac{y}{2} \right)_0^1 dy$$

$$= 2 \int_0^1 \left(x + \frac{1}{2} + \frac{1}{2} \right) dx = 2 \int_0^1 (x + 1) dx = 2 \left(\frac{x^2}{2} + x \right)_0^1$$

$$= 2 \left[\frac{1}{2} + 1 \right] = \frac{3 \times 2}{2} = 3.$$

Q5: Evaluate $\iiint A dv$ where $A = 2xz\mathbf{i} - x\mathbf{j} + y\mathbf{k}$, where v is bounded by surface $x=0, y=0, z=x^2, z=4$ and $x=2, y=6$.

Sol:

$$\begin{aligned} \iiint A dv &= \iiint (2xz\mathbf{i} - x\mathbf{j} + y\mathbf{k}) dv \\ &= \mathbf{i} \int_0^2 \int_0^6 \int_{x^2}^4 2xz \, dz \, dy \, dx - \mathbf{j} \int_0^2 \int_0^6 \int_{x^2}^4 x \, dz \, dy \, dx + \mathbf{k} \int_0^2 \int_0^6 \int_{x^2}^4 y \, dz \, dy \, dx \\ &= \mathbf{i} \int_0^2 \int_0^6 \frac{2x(z^2)}{2} \Big|_{x^2}^4 \, dy \, dx - \mathbf{j} \int_0^2 \int_0^6 x(z) \Big|_{x^2}^4 \, dy \, dx + \mathbf{k} \int_0^2 \int_0^6 y(z) \Big|_{x^2}^4 \, dy \, dx \\ &= \mathbf{i} \int_0^2 \int_0^6 (16x \, dy \, dx - \frac{1}{15} x^5) \, dy \, dx - \mathbf{j} \int_0^2 \int_0^6 (4x \, dy \, dx - \frac{1}{13} x^3) \, dy \, dx + \mathbf{k} \int_0^2 \int_0^6 (4y^2 = x^2 y^2) \, dy \, dx \\ &= \left[\mathbf{i} \int_0^2 (16xy - x^5 y) \Big|_0^6 \, dx - \mathbf{j} \int_0^2 (4xy - x^3 y) \Big|_0^6 \, dx + \mathbf{k} \int_0^2 \left(\frac{4y^3}{3} - \frac{x^2 y^3}{3} \right) \Big|_0^6 \, dx \right] \\ &= \mathbf{i} \int_0^2 (96x - 16x^5) \, dx - \mathbf{j} \int_0^2 (24x - 6x^3) \, dx + \mathbf{k} \int_0^2 (864x - \frac{216x^3}{3}) \, dx \\ &= \mathbf{i} \left[\frac{96x^2}{2} - \frac{16x^6}{6} \right]_0^2 - \mathbf{j} \left[\frac{24x^2}{2} - \frac{6x^4}{4} \right]_0^2 + \mathbf{k} \left[\frac{864x^2}{2} - \frac{216x^3}{3} \right]_0^2 \\ &= \mathbf{i} (48(4) - \frac{8(64)}{3}) - \mathbf{j} (12(4) - \frac{3(16)}{1}) + \mathbf{k} \left(\frac{864 \times 2}{3} - \frac{1728}{3} \right) \\ &= \frac{(576 - 512)}{3} \mathbf{i} - (48 - 24) \mathbf{j} + \frac{1}{3} (1728 - 576) \mathbf{k} \end{aligned}$$

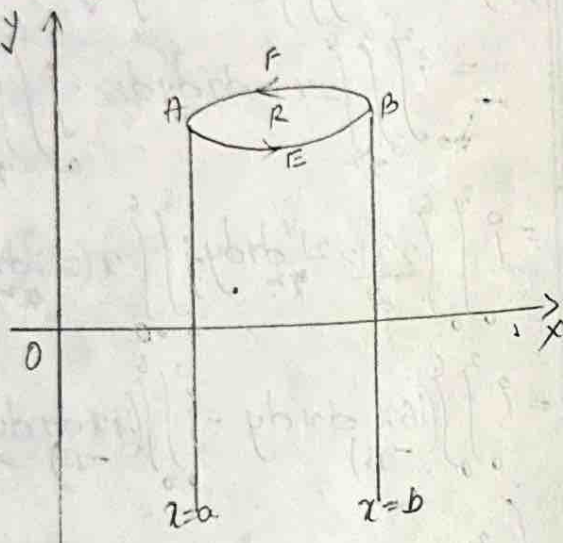
$$= (192 - 64)\hat{i} - (24)\hat{j} + K\left(\frac{1152}{3}\right) = 128\hat{i} - 24\hat{j} + 384K$$

$$\iint_{\text{AN}} = 128\hat{i} - 24\hat{j} + 384K$$

GREEN'S, GAUSS DIVERGENCE & STOKE'S

THEOREM AND APPLICATIONS

GREEN'S THEOREM:



Statement:

If R is a closed region in xy -plane bounded by a simple closed curve and M and N be continuous of x and y having continuous derivatives in R , then

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

where 'C' travels in +ve direction.

Q1: Apply Green's Theorem to evaluate $\oint_C (2xy - y^2)dx + (x^2 + y^2)dy$, where 'C' is a closed curve bounded by $y = x^2$ and $y^2 = x$.

Sol: Given, $\oint_C (2xy - y^2)dx + (x^2 + y^2)dy$

By Green's theorem, $\oint_C Mdx + Ndy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right] dx dy$

Where $M = 2xy - y^2$, $N = x^2 + y^2$

$$\frac{\partial M}{\partial y} = 2x - 2y \quad \frac{\partial N}{\partial x} = 2x$$

$$\oint_C (2xy - y^2)dx + (x^2 + y^2)dy = \iint_R -(2x - 2x + 2y) dx dy$$

$$= \iint_R 2y dx dy$$

$$= \int_0^1 \int_{\sqrt{y}}^{y^2} 2y dx dy = \int_0^1 \left[\frac{2xy}{x} \right]_{\sqrt{y}}^{y^2} dy$$

$$= \int_0^1 (y^2 - y) dy = \left[\frac{y^3}{3} - \frac{y^2}{2} \right]_0^1$$

$$= \left(\frac{1^3}{3} - \frac{1^2}{2} \right) - \left(\frac{0^3}{3} - \frac{0^2}{2} \right) = \frac{1}{3} - \frac{1}{2} = \frac{2-3}{6} = -\frac{1}{6}$$

$$y = x^2, y^2 = x$$

$$y = \sqrt{x}$$

$$x^2 = \sqrt{x}$$

$$x^2 = \sqrt{x} \Rightarrow 0$$

$$\sqrt{x}(x^{3/2} - 1) = 0$$

$$x = 1, x = 0$$

Q2: Apply Green's Theorem to evaluate $\oint_C x^2 y dx + y^3 dy$, where 'C' is a closed path formed by $y = x$ and $y = x^3$ from (0,0) to (1,1).

Sol: Given, $\oint_C x^2 y dx + y^3 dy$

By Green's Theorem, $\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$$M = x^2 y$$

$$\frac{\partial M}{\partial y} = x^2$$

$$N = y^3$$

$$\frac{\partial N}{\partial x} = 0$$

$$= (x^2)^2$$

$$\begin{aligned}
 \oint x^2 y dx + y^3 dy &= \int_0^1 \int_x^{x^3} (-x^2) dx dy \quad \begin{array}{l} y=x, y=x^3 \\ x-x^3=0 \\ x(x-x^2)=0 \\ x=0, x=1 \end{array} \\
 &= \int_0^1 -x^2 \left(y \right)_x^{x^3} dx = - \int_0^1 (x^2)(x^3-x) dx \\
 &= - \int_0^1 (x^5 - x^3) dx = - \int_0^1 (x^3 - x^5) dx = \left(\frac{x^4}{4} - \frac{x^6}{6} \right)_0^1 \\
 &= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}
 \end{aligned}$$

Q3: Evaluate $\oint (xy^2 + 2xy) dx + x^2y dy$ where 'C' is the boundary of the region enclosing $y^2 = 4x$ and $x = 3$.

Sol: Given, $\oint (xy^2 + 2xy) dx + x^2y dy$.

By Green's Theorem, $\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$$M = xy^2 + 2xy$$

$$N = x^2y$$

$$\frac{\partial M}{\partial y} = 2xy + 2x$$

$$\frac{\partial N}{\partial x} = 2xy$$

$$\oint (xy^2 + 2xy) dx + x^2y dy = \iint_R (2xy - 2xy - 2x) dx dy = \iint_R (-2x) dx dy$$

$$= \int_0^3 \int_{y^2/4}^3 (-2x) dx dy = \int_0^3 (-2xy) \Big|_{y^2/4}^3 dy = \int_0^3 (-8x^2 + 8x^2) dy$$

$$= \int_0^3 -16x^2 dy = -16 \left(\frac{y^3}{3} \right)_0^3 = \frac{-16 \times 27}{3} = -144$$

Ensemble of points (x,y) of the plane
is a subset of the plane (x,y).
(0,0) point is not included (0,0).

Let $f(x,y) = [x^2 + y^2]^{1/2}$.
Then $f(x,y) = \sqrt{x^2 + y^2}$.
The function $f(x,y)$ is continuous at (0,0).
The function $f(x,y)$ is differentiable at (0,0).

The function $f(x,y)$ is continuous at (0,0).
The function $f(x,y)$ is differentiable at (0,0).

The function $f(x,y)$ is continuous at (0,0).
The function $f(x,y)$ is differentiable at (0,0).

The function $f(x,y)$ is continuous at (0,0).
The function $f(x,y)$ is differentiable at (0,0).

The function $f(x,y)$ is continuous at (0,0).
The function $f(x,y)$ is differentiable at (0,0).

Q4: Evaluate by Green's Theorem $\oint_C e^{-x} [\sin y dx + \cos y dy]$ where 'C' is rectangular vertices enclosing $(0,0)$, $(\pi,0)$, $(\pi, \pi/2)$, $(0, \pi/2)$.

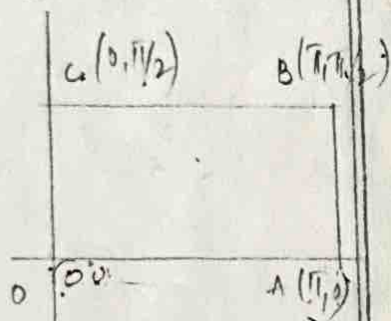
Sol: Given, $\oint_C e^{-x} [\sin y dx + \cos y dy]$.

By Green's Theorem, $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$$M = e^{-x} \sin y \quad N = +e^{-x} \cos y$$

$$\frac{\partial N}{\partial x} = -e^{-x} \cos y$$

$$\frac{\partial M}{\partial y} = e^{-x} \cos y$$



$$\oint_C e^{-x} [\sin y dx + \cos y dy] = \int_0^{\pi} \int_0^{\pi/2} -2e^{-x} \cos y dx dy$$

$$= -2 \int_0^{\pi} e^{-x} (\sin y)_{0}^{\pi/2} dx = -2 \int_0^{\pi} e^{-x} dx$$

$$= -2 \left[\frac{e^{-x}}{-1} \right]_0^{\pi} = +2 [e^{-\pi} - 1]$$

$$= 2 (e^{-\pi} - 1).$$

Q5: Evaluate by Green's Theorem $\oint_C (\cos x \sin y - xy) dx + (\sin x \cos y) dy$, where 'C' is a circle $x^2 + y^2 = 1$.

Sol:

Given, $\oint (\cos x \sin y - xy) dx + (\sin x \cos y) dy$

By Green's Theorem, $\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$M = \cos x \sin y - xy$

$N = \sin x \cos y$

$\frac{\partial M}{\partial y} = \cos x \cos y - x$

$\frac{\partial N}{\partial x} = \cos x \cos y$

$\oint (\cos x \sin y - xy) dx + (\sin x \cos y) dy = \iint_R (\cos x \cos y - \cos x \cos y + x) dx dy$
 $= \iint_R x dx dy$

$= \int_0^1 \int_0^{\sqrt{1-y^2}} x dx dy = \int_0^1 x(y) \Big|_0^{\sqrt{1-y^2}} dy$

$x^2 + y^2 = 1$

$x=0, y=1$

$y = \sqrt{1-x^2}$

$= \int_0^1 x(\sqrt{1-x^2}) dx$

put $x = \sin \theta$

$dx = \cos \theta d\theta$

Limits: x 1 0

$\sin \theta$ $\pi/2$ 0

$\int_0^{\pi/2} \sin \theta \cdot \cos \theta \cdot \cos \theta d\theta = \int_0^{\pi/2} \sin \theta \cos^2 \theta d\theta$

Formula: If $m=1$, then $\int_0^{\pi/2} \sin x \cos^n x dx = \frac{1}{n+1}$

$= \frac{1}{2+1} = \frac{1}{3}$

$\int e^{ax} dx = \frac{e^{ax}}{a}$

Q6: Verify Green's Theorem for $\int (3x^2 - 8y^2) dx + (4y - 6xy) dy$ where 'C' is the region bounded by a triangle $x=0, y=0, x+y=1$.

Sol: Given, $\int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$ ✓

By Green's Theorem, $\int_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$$\int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy = \iint_R \left(\frac{\partial}{\partial x} (4y - 6xy) - \frac{\partial}{\partial y} (3x^2 - 8y^2) \right) dx dy$$

$$\text{L.H.S} = \int_C (3x^2 - 8y^2) dx + (4y - 6xy) dy$$

Along OA: $y=0, x:(0 \rightarrow 1)$
 $dy=0$

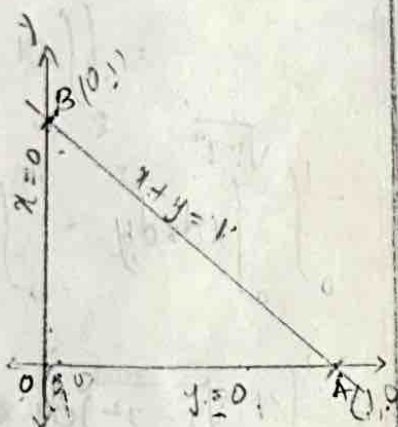
$$\int_{OA} (3x^2 - 8y^2) dx + (4y - 6xy) dy = \int_0^1 3x^2 dx$$

$$= \frac{3(x^3)}{3} \Big|_0^1 = 1$$

Along AB: $y=1-x \Rightarrow x=1-y, y:(0 \rightarrow 1)$
 $dy = -dx, dx = -dy$

$$\int_{AB} (3x^2 - 8y^2) dx + (4y - 6xy) dy = \int_0^1 (3(1-y)^2 - 8y^2)(-dy) + 4y - 6y(1-y) dy$$

$$= \int_0^1 [3(1-y)^2 + 8y^2 + 4y - 6y(1-y)] dy$$



$$x+y=1$$

$$y=1-x$$

x	0	1
y	1	0

$$= \int_0^1 -(3(1+y^2-2y) - 8y^2) dy + (4y - 6y + 6y^2) dy$$

$$= \int_0^1 (-3y^2 + 6y - 3 + 4y - 6y + 6y^2) dy$$

$$= \int_0^1 (11y^2 + 4y - 3) dy = \frac{11}{3}(y^3)_0^1 + \frac{4}{2}(y^2)_0^1 - 3(y)_0^1$$

$$= \frac{11}{3} + 2 - 3 = \frac{11}{3} - 1 = \frac{8}{3}$$

Along OB $x=0$ $y:(1 \rightarrow 0)$
 $dx=0$

$$\int_{OB} (3x^2 - 8y^2) dx + (4y - 6xy) dy = \int_1^0 4y dy = 2(y^2)_1^0 = -2$$

$$\therefore \text{L.H.S} = \int_{OAB} (3x^2 - 8y^2) dx + (4y - 6xy) dy = \frac{8}{3} + 1 - 2 = \frac{5}{3}$$

$$\text{R.H.S:} \iint_R \left[\frac{\partial}{\partial x} (4y - 6xy) - \frac{\partial}{\partial y} (3x^2 - 8y^2) \right] dx dy$$

$$= \iint_R (-6y + 16y) dx dy = \int_0^1 \int_0^{1-x} 10y dx dy = \int_0^1 \frac{10(y^2)}{2} dx$$

$$= 5 \int_0^1 (1-x)^2 dx = 5 \int_0^1 (1 + x^2 - 2x) dx = 5 \left[x + \frac{x^3}{3} - \frac{2x^2}{2} \right]_0^1$$

$$= 5 \left[1 + \frac{1}{3} - 1 \right] = \frac{5}{3}.$$

$$L \cdot H \cdot S = R \cdot H \cdot S$$

$\therefore$ Green's Theorem is Verified.

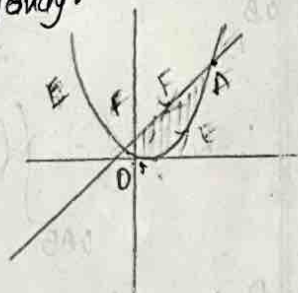
Q7: Verify Green's Theorem for plane $\oint (xy+y^2) dx + x^2 dy$ where 'C' is the closed curve for the region bounded by $y=x$ and $y=x^2$.

Sol: Given, $\oint (xy+y^2) dx + x^2 dy$

By Green's theorem, $\oint M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$

$$\oint (xy+y^2) dx + x^2 dy = \iint \left(\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y} (xy+y^2) \right) dx dy.$$

Along DEA: $y=x^2$ $x: (0 \rightarrow 1)$
 $dy = 2x dx$



$$\begin{aligned} \int_{DEA} (xy+y^2) dx + x^2 dy &= \int_0^1 (x \cdot x^2 + x^4) dx + (x^2 \cdot 2x) dx \\ &= \int_0^1 (x^3 + x^4 + 2x^3) dx = \int_0^1 (3x^3 + x^4) dx = \left(\frac{3x^4}{4} + \frac{x^5}{5} \right)_0^1 \\ &= \frac{19}{20}. \end{aligned}$$

Along AFO: $y=x$ $x:(1 \rightarrow 0)$

$$dy = dx$$

$$\int_{\text{AFO}} (xy + y^2) dx + x^2 dy = \int_1^0 (x \cdot x + x^2 \cdot 1) dx = \int_1^0 3x^2 dx = \left(\frac{3x^3}{3} \right)_1^0$$

$$= (x^3)_1^0 = -1$$

$$\text{L.H.S} = \int (xy + y^2) dx + x^2 dy = \frac{19}{20} - 1 = \frac{1}{20}$$

$$\text{R.H.S} = \iint \left(\frac{\partial}{\partial x} (x^2) - \frac{\partial}{\partial y} (xy + y^2) \right) dx dy$$

$$= \int_0^1 \int_x^{x^2} (2x - 1 - 2y) dx dy = \int_0^1 \left(xy - \frac{2y^2}{2} \right)_x^{x^2} dx$$

$$= \int_0^1 \left((x^3 - x^2) - (x^4 - x^2) \right) dx = \int_0^1 (x^3 - x^4 + x^2) dx$$

$$= \int_0^1 (x^3 - x^4) dx = \left(\frac{x^4}{4} - \frac{x^5}{5} \right)_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}$$

$$\therefore \text{L.H.S} = \text{R.H.S}$$

$\therefore$ Green's Theorem is Verified.

Q8:

$\int_C (x^2 - 2xy) dx + (x^2y + 3) dy$, Verify Green's Theorem where C is the boundary region bounded $y^2 = 8x$, $x = 2$.

Sol:

$$\text{Given, } \int_C (x^2 - 2xy) dx + (x^2y + 3) dy$$

By Green's Theorem,

$$\int Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

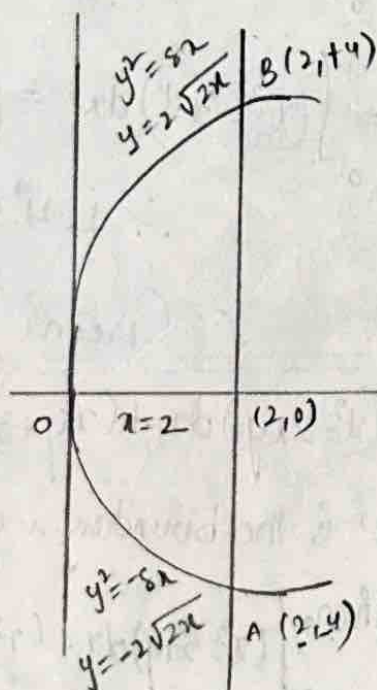
$$\int (x^2 - 2xy)dx + (x^2y + 3)dy = \iint_R \left(\frac{\partial}{\partial x} (x^2y + 3) - \frac{\partial}{\partial y} (x^2 - 2xy) \right) dx dy$$

$$\text{R.H.S} = \iint_{-2\sqrt{x}}^{2\sqrt{x}} (2xy + 2x) dx dy = \int_0^2 \left(\frac{2xy^2}{2} + 2xy \right) dx \Big|_{-2\sqrt{x}}^{2\sqrt{x}}$$

$$= \int_0^2 \left[x(2\sqrt{x})^2 + 2x(2\sqrt{x}) \right] - \left[x(-2\sqrt{x})^2 + 2x(-2\sqrt{x}) \right] dx$$

$$= \int_0^2 (8x^2 + 4\sqrt{2}x^{3/2} - 8x^2 + 4\sqrt{2}x^{3/2}) dx = \int_0^2 8\sqrt{2}x^{3/2} dx$$

$$= 8\sqrt{2} \left(\frac{x^{5/2}}{5/2} \right)_0^2 = \frac{8 \times \sqrt{2} \times 2 \times \sqrt{25}}{5} = \frac{128}{5}$$



Along AO: BO: $y = 2\sqrt{2}\sqrt{x}$
 $dy = \frac{1}{\sqrt{2}} dx = \frac{1}{\sqrt{2}} \sqrt{x} dx$

$$\int (x^2 - 2xy) dx + (x^2y + 3) dy = \int_0^2 x^2 - 2x(2\sqrt{2}\sqrt{x}) dx$$

$$+ (x^2(1\sqrt{2}x + 3)(\sqrt{2}x^{-1/2}) dx$$

$$= \int x^2 - 4\sqrt{2}x^{3/2} + 4x^2 + 3(2x^{-1/2}) dx$$

$$= \int_0^2 \left(\frac{5x^3}{3} - 4\sqrt{2} \frac{x^{5/2}}{5/2} + 3\sqrt{2} \frac{x^{1/2}}{1/2} \right) dx$$

$$= - \left[5 \times \frac{8}{3} - \frac{4 \times 2\sqrt{2} \times 4\sqrt{2}}{5} + 6(2) \right]$$

$$= \frac{40}{3} - \frac{64}{5} + 12 = -\frac{188}{15}$$

Along AB:

$x = 2$
 $dx = 0$ $y = -4 \rightarrow 4$

$$\int (x^2 - 2xy) dx + (x^2y + 3) dy = \int_{-4}^4 (4y + 3) dy$$

$$= \left(\frac{4y^2}{2} + 3y \right)_{-4}^4 = 24$$

$$L.H.S = \int_{OAB} (x^2 - 2xy) dx + (x^2y + 3) dy$$

$$= \frac{212}{15} - \frac{188}{15} + 24 = \frac{128}{5}$$

$$\therefore LHS = RHS$$

$\therefore$ Green's theorem is verified.

$$\text{L.H.S} = \iint (x^2 - 2xy) dx + (x^2y + 3) dy$$

Along OA: $y = -2\sqrt{2}x$

$$dy = -2\sqrt{2} \times \frac{1}{2\sqrt{x}} dx$$

$$x=0 \rightarrow 2$$

$$= \int_0^2 (x^2 - 2x(-2\sqrt{2}x)) dx + (x^2(-2\sqrt{2}x) + 3)(-\sqrt{2}x^{-1/2}) dx$$

$$= \int_0^2 (x^2 + 4\sqrt{2}x^{3/2} + 4x^2 - 3\sqrt{2}x^{-1/2}) dx$$

$$= \int_0^2 (5x^2 + 4\sqrt{2}x^{3/2} - 3\sqrt{2}x^{-1/2}) dx$$

$$= \left[\frac{5x^3}{3} + 4\sqrt{2} \cdot \frac{2}{5} x^{5/2} - 3\sqrt{2}(2)x^{-1/2} \right]_0^2$$

$$= \left[\frac{5x^3}{3} + \frac{8\sqrt{2}}{5} x^{5/2} - 6\sqrt{2}x^{-1/2} \right]_0^2$$

$$= \left[\frac{5(2)^3}{3} + \frac{8}{5}\sqrt{2}x^{5/2} - 12\sqrt{2}x^{-1/2} \right]$$

$$= \frac{40}{3} + \frac{64}{5} - 12 = \frac{40}{3}$$

$$= \frac{40}{3} + \frac{64-60}{5}$$

$$= \frac{40}{3} + \frac{4}{5} = \frac{200+12}{15} = \frac{212}{15}$$

Q9: Verify Green's theorem in plane $\phi(x^2 - xy^3)$ where 'C' is the square with the vertices (2,2) (0,2).

Sol: By Green's Theorem,

$$\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\oint (x^2 - xy^3) dx + (y^2 - 2xy) dy =$$

$$\iint_R \left(\frac{\partial}{\partial x} (y^2 - 2xy) - \frac{\partial}{\partial y} (x^2 - xy^3) \right) dx dy$$

$$\text{L.H.S: } \oint (x^2 - xy^3) dx + (y^2 - 2xy) dy$$

Along OA: $y=0$ $x:(0 \rightarrow 2)$
 $dy=0$

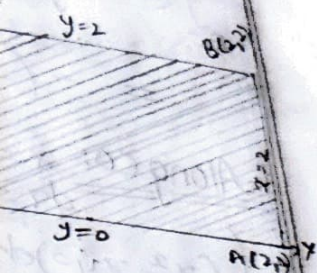
$$\int_{OA} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_0^2 (x^2 - 0) dx = \int_0^2 x^2 dx = \left(\frac{x^3}{3} \right)_0^2 = \frac{8}{3}$$

Along AB: $x=2$ $y:(0 \rightarrow 2)$
 $dx=0$

$$\int_{AB} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_0^2 (y^2 - 4y) dy = \left(\frac{y^3}{3} - \frac{4y^2}{2} \right)_0^2 = \frac{8}{3} - 2(4) = -\frac{10}{3}$$

$$)dx + (y^2 - 2xy)dy$$

$$(0,0) \quad (2,0)$$



→ (A)

$$xy^3)dx + 0$$

$$8/3$$

$$= 4y)dy$$

$$= \frac{8}{3} - 8$$

Along BC: $y=2$ $x: (2 \rightarrow 0)$
 $dy=0$

$$\int_{BC} (x^2 - xy^3) dx + 0 = \int_2^0 (x^2 - 8x) dx = \left(\frac{x^3}{3} - 8 \left(\frac{x^2}{2} \right) \right)_2^0$$
$$= - \left(\frac{8}{3} - 4(4) \right) = 16 - \frac{8}{3}$$

Along CO: $x=0$ $y: (2 \rightarrow 0)$
 $dx=0$

$$\int_{CO} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \int_2^0 y^2 dy = \left(\frac{y^3}{3} \right)_2^0 = -\frac{8}{3}$$

Along OABC: $\int_{OABC} (x^2 - xy^3) dx + (y^2 - 2xy) dy = \frac{8}{3} + \frac{8}{3} - 8 + 16 - \frac{8}{3} - \frac{8}{3}$

$$= \frac{16}{3} - \frac{16}{3} + 8 = 8$$

R.H.S: $\iint_R \left(\frac{\partial}{\partial x} (y^2 - 2xy) - \frac{\partial}{\partial y} (x^2 - xy^3) \right) dx dy$

$$= \iint_R (-2y + 3y^2x) dx dy = \int_0^2 \int_0^2 (3xy^2 - 2y) dx dy$$

$$= \int_0^2 \left(\frac{3xy^3}{3} - \frac{2y^2}{2} \right)_0^2 dy = \int_0^2 (8y^2 - 4y) dy = \left(\frac{8y^3}{3} - 4y^2 \right)_0^2$$

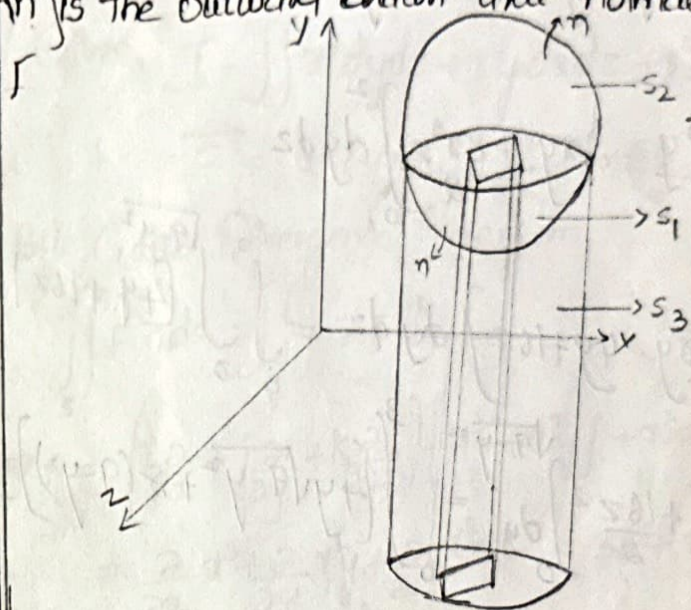
$$= 4(4) - 4(2) = 16 - 8 = 8$$

$$\therefore L.H.S = R.H.S$$

$\therefore$ Hence Green's Theorem is Verified.

GAUSS DIVERGENCE THEOREM —

STATEMENT: If $\vec{F}$ is a vector point-function having continuous first order partial derivative a closed surface enclosing a volume V , then $\iint_S \vec{F} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{F} dv$, where $\vec{n}$ is the outward drawn unit normal to the surface.



Q1: Evaluate $\iint_S \vec{F} \cdot d\vec{s}$ where $\vec{F} = 2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k}$ where 'S' is the surface $y^2 + z^2 = 9$ and $x = 2$, in the first octant.

Sol: Given, $\vec{F} = 2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k}$

By Gauss Divergence Theorem,

$$\iint_S \vec{F} \cdot d\vec{s} = \iiint_V \nabla \cdot \vec{F} dv$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (2x^2y\vec{i} - y^2\vec{j} + 4xz^2\vec{k}) \\ &= 4xy - 2y + 8xz \end{aligned}$$

$$\iint_S F \cdot \mathbf{n} \, ds = \iiint_V (4xy - 2y + 8xz) \, dx \, dy \, dz$$

$$\begin{aligned} y^2 + z^2 &= 9 \\ z^2 &= 9 - y^2 \\ z &= \sqrt{9 - y^2} \end{aligned}$$

$$= \int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-y^2}} (4xy - 2y + 8xz) \, dx \, dy \, dz$$

$$= \int_0^3 \int_0^{\sqrt{9-y^2}} \left[\frac{4x^2y}{2} - 2xy + \frac{8x^2z}{2} \right]_0^{\sqrt{9-y^2}} dy \, dz$$

$$= \int_0^3 \int_0^{\sqrt{9-y^2}} [8y - 4y + 16z] dy \, dz = \int_0^3 \int_0^{\sqrt{9-y^2}} [4y + 16z] dy \, dz$$

$$= \int_0^3 \left[4yz + \frac{16z^2}{2} \right]_0^{\sqrt{9-y^2}} dy = \int_0^3 [4y\sqrt{9-y^2} + 8(9-y^2)] dy$$

$$\text{Put } 9 - y^2 = t^2$$

$$\text{Limits: } \begin{array}{ccc} y & 0 & 3 \\ t\sqrt{9-y^2} & 3 & 0 \end{array}$$

$$-2y \, dy = 2t \, dt$$

$$y \, dy = -dt \cdot t$$

$$= \int_0^3 4y\sqrt{9-y^2} \, dy + 8 \int_0^3 (9-y^2) \, dy$$

$$= \int_3^0 4 \cdot t(t) \, dt + 8 \left[\int_0^3 9 \, dy - \int_0^3 y^2 \, dy \right]$$

$$= -4 \int_3^0 t^2 \, dt + 8 \left[(9y)_0^3 - \left(\frac{y^3}{3} \right)_0^3 \right]$$

$$= -4 \left(\frac{t^3}{3} \right)_3^0 + 8 \left[9(3) - \frac{27}{3} \right] = -4(0-9) + 8(27-9) \\ = 36 + (8 \times 18) = 36 + 144 \\ = 180.$$

Q2: Apply Gauss-Divergence theorem to evaluate $I = \iint_S x^3 dy dz + x^2 y dz dx + x^2 z dx dy$, where 'S' is the closed surface bounded by the planes $z=0$, $z=b$ and the cylinder $x^2 + y^2 = a^2$.

Sol: Given, $I = \iint_S x^3 dy dz + x^2 y dz dx + x^2 z dx dy$

$$\vec{F} = x^3 \hat{i} + x^2 y \hat{j} + x^2 z \hat{k}$$

By Gauss Divergence theorem,

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \nabla \cdot \vec{F} dv$$

$$\begin{aligned} \nabla \cdot \vec{F} &= \left(\hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z} \right) \cdot (x^3 \hat{i} + x^2 y \hat{j} + x^2 z \hat{k}) \\ &= \frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} x^2 y + \frac{\partial}{\partial z} x^2 z = 3x^2 + x^2 + x^2 = 5x^2 \end{aligned}$$

$$I = \iint_S x^3 dy dz + x^2 y dz dx + x^2 z dx dy = \iiint_V 5x^2 dx dy dz \rightarrow (A)$$

$$= \int_{z=0}^b \int_{y=0}^a \int_{x=0}^{\sqrt{a^2-y^2}} 5x^2 dx dy dz = \int_{z=0}^b \int_{y=0}^a \left(\frac{5x^3}{3} \right)_0^{\sqrt{a^2-y^2}} dy dz$$

$$= \int_{z=0}^b \int_{y=0}^a \frac{5}{3} (a^2 - y^2)^{3/2} dy dz = \int_{y=0}^a \frac{5}{3} (a^2 - y^2)^{3/2} [z]_0^b dy$$

$$= \frac{5b}{3} \int_{y=0}^a (a^2 - y^2)^{3/2} dy$$

Put $y = a \sin t$
 $dy = a \cos t dt$

limits:
 $y = 0 \rightarrow a$
 $t = \sin^{-1}(y/a)$
 l. limit $= \sin^{-1}(0) = 0$
 u. limit $= \sin^{-1}(a/a) = \pi/2$

$$= \frac{5b}{3} \int_0^{\pi/2} \sqrt{(a^2 - a^2 \sin^2 t)} \cdot a \cos t dt$$

$$= \frac{5b}{3} \int_0^{\pi/2} a^2 \cdot a \cos^3 t \cdot \cos t dt$$

$$= \frac{5b}{3} a^4 \int_0^{\pi/2} \cos^4 t dt = \frac{5b}{3} \times a^4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{5a^4 b \pi}{16}$$

Q3. If $\vec{F} = 2xy\mathbf{i} + yz^2\mathbf{j} + xz\mathbf{k}$ and 'S' is a rectangular parallelepiped bounded by $x=0, y=0, z=0, x=2, y=1, z=3$. Evaluate $\iint_S \vec{F} \cdot \vec{n} ds$.

Sol: Given, $\vec{F} = 2xy\mathbf{i} + yz^2\mathbf{j} + xz\mathbf{k}$

According to Gauss-Divergence theorem,

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \nabla \cdot \vec{F} dv$$

$$\nabla \cdot \vec{F} = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) (2xy\mathbf{i} + yz^2\mathbf{j} + xz\mathbf{k})$$

$$= 2y + z^2 + 1$$

$$\iiint_V 2xy\mathbf{i} + yz^2\mathbf{j} + xz\mathbf{k} = \iiint_V (2 + 2y + z^2) dv$$

$$\begin{aligned}
 &= \int_{x=0}^2 \int_{y=0}^1 \int_{z=0}^3 (x+2y+z^2) dx dy dz = \int_{x=0}^2 \int_{y=0}^1 (3x+6y+9) dx dy \\
 &= \int_{x=0}^2 \left(3xy + \frac{6y^2}{2} + 9y \right)_0^1 dx = \int_{x=0}^2 (3x+3+9) dx \\
 &= \left(\frac{3x^2}{2} + 12x \right)_0^2 = \frac{3}{2}(4) + 12(2) = 30.
 \end{aligned}$$

Q4: Verify Gauss-Jordan Divergence theorem for $\vec{F} = (x^3yz)\mathbf{i} - 2x^2y\mathbf{j} + z\mathbf{k}$ taken over the surface $0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a$ of the cube.

Sol: Given, $\vec{F} = (x^3yz)\mathbf{i} - 2x^2y\mathbf{j} + z\mathbf{k}$

By Gauss-Divergence Theorem, $\iint_V \vec{F} \cdot \vec{n} ds = \iiint_V \nabla \cdot \vec{F} dv$

$$\iint_S ((x^3yz)\mathbf{i} - 2x^2y\mathbf{j} + z\mathbf{k}) \cdot \vec{n} ds = \iiint_V \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z} \right) \cdot ((x^3yz)\mathbf{i} - 2x^2y\mathbf{j} + z\mathbf{k}) dv$$

$$\text{R.H.S} = \int_0^a \int_0^a \int_0^a \left[\frac{\partial}{\partial x} (x^3yz) - \frac{\partial}{\partial y} (2x^2y) + \frac{\partial}{\partial z} (z) \right] dv$$

$$\int_0^a \int_0^a \int_0^a (3x^2 - 2x^2 + 1) dx dy dz = \int_0^a \int_0^a \int_0^a (x^2 + 1) dx dy dz$$

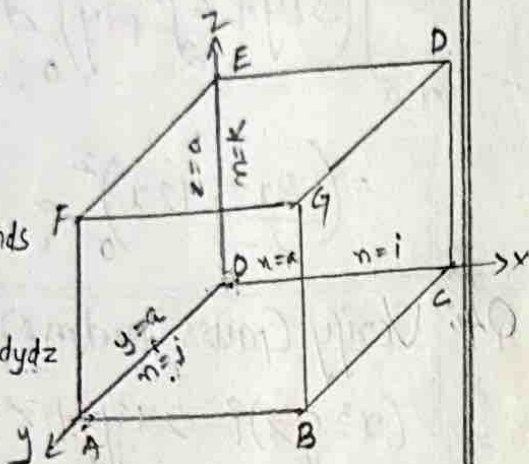
$$\int_0^a \int_0^a (x^2+1) dx dy (z)_0^a = a \int_0^a (x^2+1) (y)_0^a dx$$

$$= a^2 \int_0^a (x^2+1) dx = a^2 \left[\frac{x^3}{3} + x \right]_0^a = a^2 \left(\frac{a^3}{3} + a \right)$$

$$= \frac{a^5}{3} + a^3 \longrightarrow \textcircled{A}.$$

L.H.S: $\iiint ((x^3-yz)i - 2x^2y j + zk) \cdot n ds$

→ Surface BCDG: $x=a, n=i, ds=dydz$



$$\iiint_S ((x^3-yz)i - 2x^2y j + zk) \cdot n ds$$

$$= \int_0^a \int_0^a ((a^3-yz)i - 2a^2y j + zk) \cdot i dy dz = \int_0^a \int_0^a (a^3-yz) dy dz$$

$$= \int_0^a \left(a^3z - y \frac{z^2}{2} \right)_0^a dy = \int_0^a \left(a^4 - \frac{y a^2}{2} \right) dy = \left(a^4 y - \frac{y^2}{2} \cdot \frac{a^2}{2} \right)_0^a$$

$$= a^5 - \frac{a^4}{4} \longrightarrow \textcircled{1}$$

→ Surface OAEF: $x=0, n=-i, ds=dydz$.

$$\iiint ((x^3-yz)i - 2x^2y j + zk) \cdot n ds = \iiint ((0-yz)i - 0 \cdot j + z \cdot k) \cdot (-i) ds$$

$$= \int_0^a \int_0^a yz dy dz = \int_0^a \left(\frac{y^2}{2} z \right)_0^a dy = \frac{a^2}{2} \int_0^a y dy$$

$$= \frac{a^2}{2} \left(\frac{y^2}{2} \right)_0^a = \frac{a^2}{2} \cdot \frac{a^2}{2} = \frac{a^4}{4} \longrightarrow \textcircled{2}$$

→ SURFACE ABGF: $y=a$, $n=j$, $ds=dx dz$.

$$\iiint_S ((x^3-yz)i - 2x^2y j + zk) \cdot n ds = \iint ((x^3-az)i - 2x^2a j + zk) \cdot j ds$$

$$= \int_0^a \int_0^a (-2x^2a) dx dz = -2a \int_0^a x(z)_0^a dx = -2a^2 \int_0^a x^2 dx$$

$$= -2a^2 \left(\frac{x^3}{3} \right)_0^a = \frac{-2a^5}{3} \rightarrow \textcircled{3}$$

→ SURFACE OCDE: $y=0$, $n=-j$, $ds=dx dz$

$$\iiint_S ((x^3-yz)i - 2x^2y j + zk) \cdot n ds = \iint (zk) \cdot (-j) dx dz = 0 \rightarrow \textcircled{4}$$

→ SURFACE OABC: $z=0$, $n=-k$, $ds=dx dy$

$$\iiint_S ((x^3-yz)i - 2x^2y j + zk) \cdot n ds = \iint -(2x^2y j) \cdot (-k) dx dy = 0 \rightarrow \textcircled{5}$$

→ SURFACE DEFG: $z=a$, $n=k$, $ds=dx dy$

$$\iiint_S ((x^3-yz)i - 2x^2y j + zk) \cdot n ds = \iint ((x^3-ay)i - 2x^2y j + ak) \cdot k ds$$

$$= \int_0^a \int_0^a a dx dy = a \int_0^a (y)_0^a dx = a^2 \int_0^a dx = a^2 [x]_0^a = a^3 \rightarrow \textcircled{6}$$

$$L.H.S = \iiint_S ((x^3-yz)i - 2x^2y j + zk) \cdot n ds = a^5 - \frac{a^4}{4} + \frac{a^4}{4} - 2a^5 + a^3$$

$$= \frac{3a^5 - 2a^5}{3} + a^3 = \frac{a^5}{3} + a^3 \rightarrow \textcircled{B}$$

$$L.H.S = R.H.S, \quad \iiint_S F \cdot n ds = \iiint_V \nabla \cdot F dv, \quad \therefore \text{Gauss-Divergence Theorem is Verified.}$$

Q5. Verify Divergence Theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$, taken over the cube bounded by $x=0, x=1, y=0, y=1$ and $z=0, z=1$.

Sol: By Gauss-Divergence Theorem,

$$\iint_S \vec{F} \cdot \vec{n} ds = \iiint_V \nabla \cdot \vec{F} dv$$

$$\iint_S (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) \cdot \vec{n} ds = \iiint_V \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot \vec{F} dv$$

$$\text{R.H.S} = \iiint_V \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) \cdot (4xz\vec{i} - y^2\vec{j} + yz\vec{k}) dv$$

$$= \iiint_V \left(\frac{\partial}{\partial x} (4xz) - \frac{\partial}{\partial y} (y^2) + \frac{\partial}{\partial z} (yz) \right) dx dy dz$$

$$= \int_0^1 \int_0^1 \int_0^1 (4z - 2y + y) dx dy dz = \int_0^1 \int_0^1 (4z - y) dx dy dz$$

$$= \int_0^1 \int_0^1 \left(\frac{4xz^2}{2} - yz \right)_0^1 dx dy = \int_0^1 \int_0^1 (2 - y) dx dy$$

$$= \int_0^1 \left(2y - \frac{y^2}{2} \right)_0^1 dy = \int_0^1 \left(2 - \frac{1}{2} \right) dy = \frac{3}{2} \int_0^1 1 dy = \frac{3}{2} (y)_0^1$$

$$= \frac{3}{2}$$

$$\therefore \iiint_V \nabla \cdot \vec{F} dv = \frac{3}{2} \longrightarrow \text{A}$$

R.H.S: $\iint_S \mathbf{F} \cdot \mathbf{n} \, ds$

→ SURFACE AFGB:

$x=1, \mathbf{n}=\mathbf{i}, ds=dydz$

$$\int_0^1 \int_0^1 (4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}) \cdot \mathbf{i} \, dydz$$

$$= \int_0^1 \int_0^1 (4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}) \cdot \mathbf{i} \, dydz = \int_0^1 \int_0^1 4z \, dydz$$

$$= \int_0^1 \left(\frac{4z^2}{2} \right)_0^1 dy = \int_0^1 2 \, dy = (2y)_0^1 = 2 \rightarrow \textcircled{1}$$

→ SURFACE OEDC:

$x=0, \mathbf{n}=-\mathbf{i}, ds=dydz$

$$\int_0^1 \int_0^1 (4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}) \cdot (-\mathbf{i}) \, ds = \int_0^1 \int_0^1 -(4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}) \cdot (-\mathbf{i}) \, dydz$$

$$= 0. \rightarrow \textcircled{2}$$

→ SURFACE GDEF:

$y=1, \mathbf{n}=\mathbf{j}, ds=dx \, dz$

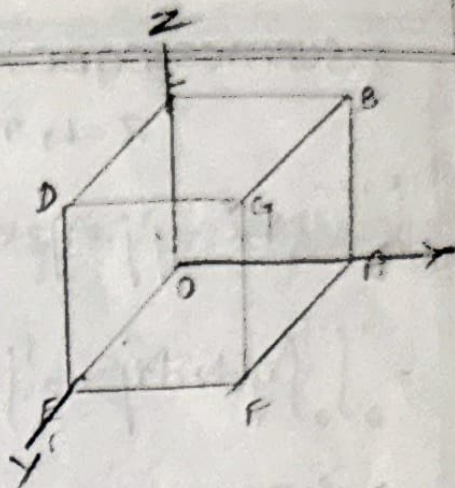
$$\int_0^1 \int_0^1 (4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}) \cdot \mathbf{j} \, ds = \int_0^1 \int_0^1 (4xz\mathbf{i} - \mathbf{j} + z\mathbf{k}) \cdot \mathbf{j} \, dx \, dz$$

$$= - \int_0^1 (z)_0^1 dz = - \int_0^1 dz = -1 \rightarrow \textcircled{3}$$

→ SURFACE OABC:

$y=0, \mathbf{n}=-\mathbf{j}, ds=dx \, dz$

$$\int_0^1 \int_0^1 (4xz\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}) \cdot (-\mathbf{j}) \, ds = \int_0^1 \int_0^1 (4xz\mathbf{i}) \cdot (-\mathbf{j}) \, dx \, dz = 0 \rightarrow \textcircled{4}$$



→ SURFACE OABC:

$$z=1, n=k, ds=dx dy$$

$$\begin{aligned} \int_0^1 \int_0^1 (xz i - y^2 j + yzk) \cdot k \, ds &= \int_0^1 \int_0^1 (xz i - y^2 j + yk) \cdot k \, dx \, dy \\ &= \int_0^1 \int_0^1 y \, dx \, dy = \int_0^1 \left(\frac{y^2}{2} \right)'_0^1 dy = \int_0^1 \frac{1}{2} dy = \frac{1}{2} y = \frac{1}{2} \rightarrow \textcircled{A} \end{aligned}$$

→ SURFACE OAFE:

$$z=0, n=-k, ds=dx dy$$

$$\int_0^1 \int_0^1 (xz i - y^2 j + yzk) \cdot (-k) \, dx \, dy = \int_0^1 \int_0^1 (-y^2 j) \cdot (-k) \, dx \, dy = 0 \rightarrow \textcircled{B}$$

$$\therefore L.H.S = \iint_S F \cdot n \, ds = \frac{1}{2} + 0 + 2 + 0 - 1 + 0 = \frac{1}{2} + 1 = \frac{3}{2} \rightarrow \textcircled{B}$$

From $\textcircled{A}$ & $\textcircled{B}$,

$$L.H.S = R.H.S$$

$$\iint_S F \cdot n \, ds = \iiint_V \nabla \cdot F \, dv$$

$\therefore$ Gauss-Divergence Theorem is Verified.

Q6: Verify Divergence Theorem for $\vec{F} = (x^2 - yz)i + (y^2 - xz)j + (z^2 - xy)k$ taken over a rectangular parallelepiped $0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$.

Sol: Given, $\vec{F} = (x^2 - yz)i + (y^2 - xz)j + (z^2 - xy)k$.

By Gauss Divergence Theorem,

$$\iint_S F \cdot n \, ds = \iiint_V \nabla \cdot F \, dv$$

$$\text{R.H.S} = \iiint \nabla \cdot \mathbf{F} \, dv,$$

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \left(i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot [(x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k] \\ &= \frac{\partial}{\partial x} (x^2 - yz) + \frac{\partial}{\partial y} (y^2 - zx) + \frac{\partial}{\partial z} (z^2 - xy) \\ &= 2x + 2y + 2z = 2(x + y + z). \end{aligned}$$

$$\iiint_V \nabla \cdot \mathbf{F} \, dv = \iiint_0^a \int_0^b \int_0^c 2(x + y + z) \, dx \, dy \, dz = 2 \int_0^a \int_0^b \left(xz + yz + \frac{z^2}{2} \right) \, dx \, dy$$

$$= 2 \int_0^a \int_0^b \left(xc + yc + \frac{c^2}{2} \right) \, dx \, dy = 2 \int_0^a \left(xcy + \frac{cy^2}{2} + \frac{c^2}{2}y \right) \Big|_0^b \, dx$$

$$= 2 \int_0^a \left(xcb + \frac{cb^2}{2} + \frac{c^2b}{2} \right) \, dx = 2 \left[\frac{x^2 cb}{2} + \frac{cb^2}{2}x + \frac{c^2b}{2}x \right]_0^a$$

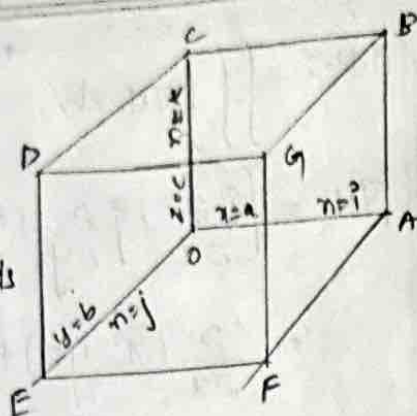
$$= 2 \left[\frac{a^2 cb}{2} + \frac{cb^2 a}{2} + \frac{c^2 ba}{2} \right] = \frac{2abc}{2} (a + b + c)$$

$$\therefore \iiint_V \nabla \cdot \mathbf{F} \, dv = abc(a + b + c) \longrightarrow \textcircled{A}$$

$$\text{R.H.S} = \iint_S \mathbf{F} \cdot \mathbf{n} \, ds = \iint_S [(x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k] \cdot \mathbf{n} \, ds$$

SURFACE DQBC:

$$z=c, n=k, ds=dx dy$$



$$\iint ((x^2-yz)i + (y^2-xz)j + (z^2-xy)k) \cdot n ds$$

$$= \int_{x=0}^a \int_{y=0}^b ((x^2-yz)i + (y^2-xz)j + (z^2-xy)k) \cdot k dx dy$$

$$= \int_{x=0}^a \int_{y=0}^b (c^2-xy) dx dy = \int_{x=0}^a \left(c^2 y - \frac{xy^2}{2} \right) \Big|_0^b dx$$

$$= \int_0^a \left(c^2 b - \frac{ab^2}{2} \right) dx = \left(c^2 bx - \frac{x^2 \cdot b^2}{2} \right) \Big|_0^a = ac^2b - \frac{b^2a^2}{4} \rightarrow \textcircled{1}$$

SURFACE ODAFE: $z=0, n=-k, ds=dx dy$

$$\iint ((x^2-yz)i + (y^2-xz)j + (z^2-xy)k) \cdot (-k) dx dy$$

$$= \int_0^a \int_0^b -(0-xy) dx dy = \int_0^a \int_0^b xy dx dy$$

$$= \int_0^a x \left(\frac{y^2}{2} \right) \Big|_0^b dx = \frac{b^2}{2} \int_0^a x dx = \left(\frac{x^2}{2} \right) \Big|_0^a \cdot \frac{b^2}{2}$$

$$= \frac{b^2a^2}{4} \rightarrow \textcircled{2}$$

SURFACE AFGH:

$$x=a, n=i, ds=dydz$$

$$\begin{aligned} & \iint ((x^2-yz)i + (y^2-xz)j + (z^2-xy)k) \cdot n \, ds \\ &= \iint ((x^2-yz)i + (y^2-xz)j + (z^2-xy)k) \cdot i \, dy \, dz \\ &= \int_0^b \int_0^c (x^2-yz) \, dy \, dz = \int_0^b \int_0^c (a^2-yz) \, dy \, dz \\ &= \int_0^b \left(a^2z - y \frac{z^2}{2} \right)_0^c \, dy = \int_0^b \left(a^2c - y \frac{c^2}{2} \right) \, dy = \left(a^2cy - \frac{y^2}{2} \frac{c^2}{2} \right)_0^b \\ &= a^2cb - \frac{b^2c^2}{4} \longrightarrow \textcircled{3} \end{aligned}$$

SURFACE OEDC:

$$x=0, n=-i, ds=dydz$$

$$\begin{aligned} & \iint ((x^2-yz)i + (y^2-xz)j + (z^2-xy)k) \cdot (-i) \, ds \\ &= \int_0^b \int_0^c -(x^2-yz) \, dy \, dz = \int_0^b \int_0^c yz \, dy \, dz \\ &= \int_0^b y \left(\frac{z^2}{2} \right)_0^c \, dy = \int_0^b y \frac{c^2}{2} \, dy = \left(\frac{y^2}{2} \frac{c^2}{2} \right)_0^b = \frac{b^2c^2}{4} \longrightarrow \textcircled{4} \end{aligned}$$

SURFACE GDEF: $y=b, n=j, ds=dx \, dz$

$$\iint ((x^2-yz)i + (y^2-xz)j + (z^2-xy)k) \cdot j \, dx \, dz$$

$$\int_0^a \int_0^c (y^2 - xz) dx dz = \int_0^a \int_0^c (b^2 - xz) dx dz = \int_0^a \left(b^2 z - \frac{xz^2}{2} \right)_0^c dz$$

$$\int_0^a \left(b^2 c - \frac{x c^2}{2} \right) dx = \left(b^2 cx - \frac{x^2}{2} \cdot \frac{c^2}{2} \right)_0^a = b^2 ca - \frac{a^2 c^2}{4} \rightarrow \textcircled{5}$$

SURFACE OABC:

$$y=0, n=-j, ds=dx dz$$

$$\iint ((x^2 - yz)i + (y^2 - xz)j + (z^2 - xy)k) \cdot (-j) ds$$

$$= \int_0^a \int_0^c - (y^2 - xz) dx dz = \int_0^a \int_0^c xz dx dz = \int_0^a \left(\frac{x^2}{2} \right)_0^c dz$$

$$= \left(\frac{x^2 c^2}{2} \right)_0^a = \frac{c^2 a^2}{4} \rightarrow \textcircled{6}$$

$$\therefore \text{L.H.S} = ac^2b - \frac{b^2 a^2}{4} + \frac{a^2 b^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{b^2 c^2}{4} + b^2 ca - \frac{a^2 c^2}{4} + \frac{a^2 c^2}{4}$$

$$= ac^2b + a^2bc + b^2ca = abc(a+b+c) \rightarrow \textcircled{B}$$

From (A) & (B), L.H.S = R.H.S

$$\iint_S F \cdot n ds = \iiint_V \nabla \cdot F dV$$

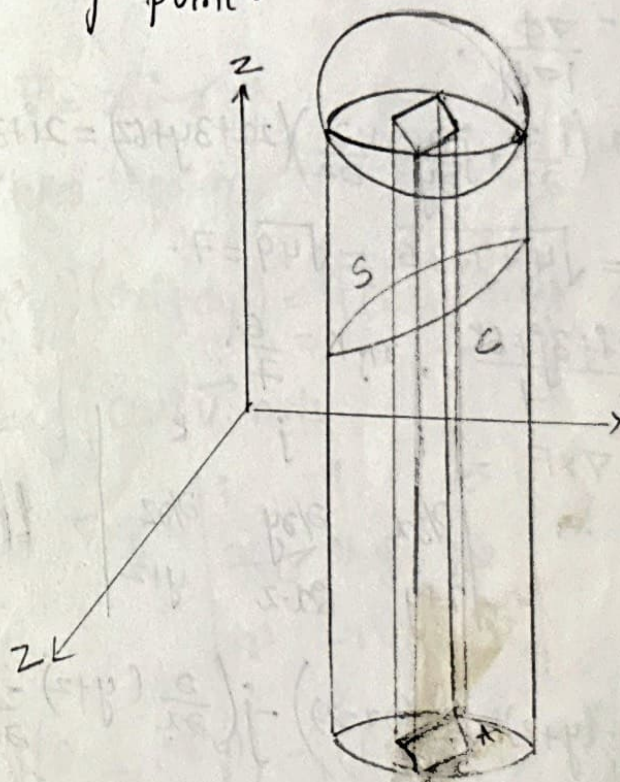
Hence Gauss - Divergence Theorem is verified.

STOKES THEOREM

STATEMENT: If "S" be an open surface bounded by closed curve "C", and $\vec{F}$ be any vector point function having continuous first order partial derivatives, then

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot \vec{n} ds = \iint_S \text{curl } \vec{F} ds,$$

where ' $\vec{n}$ ' is outward drawn unit normal to the surface at any point.



Q1: Apply Stokes Theorem to evaluate $\oint (x+y)dx + (y+z)dz + (2x-z)dy$ where 'C' is the boundary of triangle of vertices $(2,0,0), (0,3,0), (0,0,6)$.

Sol: By Stokes Theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint \text{curl } \vec{F} \cdot \vec{n} \, ds = \iint \nabla \times \vec{F}$

η = equation of plane $\phi = 2x + 3y + 6z = 0$.

$$\vec{F} = (x+y)\vec{i} + (2x-z)\vec{j} + (y+z)\vec{k}.$$

$$\vec{n} = \frac{\nabla \phi}{|\nabla \phi|},$$

$$\nabla \phi = \left(\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right) (2x + 3y + 6z) = 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$$|\nabla \phi| = \sqrt{4 + 9 + 36} = \sqrt{49} = 7.$$

$$\vec{n} = \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7}, \quad \text{and } k = \frac{6}{7}.$$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \vec{i} \left(\frac{\partial}{\partial y} (y+z) - \frac{\partial}{\partial z} (2x-z) \right) - \vec{j} \left(\frac{\partial}{\partial x} (y+z) - \frac{\partial}{\partial z} (x+y) \right)$$

$$+ \vec{k} \left(\frac{\partial}{\partial x} (2x-z) - \frac{\partial}{\partial y} (x+y) \right)$$

$$= \vec{i}(1+1) - \vec{j}(0-0) + \vec{k}(2-1) = 2\vec{i} + \vec{k}.$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint \text{curl } \vec{F} \cdot \vec{n} \, ds = \iint (2\vec{i} + \vec{k}) \cdot \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7} \cdot \frac{dx \, dy}{|\vec{n} \cdot \vec{k}|}$$

$$\iint (2i+k) \cdot (2i+3j+6k) \cdot \frac{dx dy}{(6/\sqrt{2})} = \frac{1}{\sqrt{2}} \iint (4+6) dx dy = \frac{10}{\sqrt{2}} \int_0^2 \int_0^3 dx dy$$

$$= \frac{10}{\sqrt{2}} \int_0^2 (4y)_0^3 dx = \frac{10}{\sqrt{2}} \times 3 \int_0^2 dx = \frac{10}{\sqrt{2}} \times 3 \times 2 = 10\sqrt{2}$$

Q2: Verify Stokes Theorem for $\vec{V} = x^3\vec{i} - x^2y\vec{j}$, where C is the boundary of rectangle whose sides are $x=0$, $x=3$, $y=0$, $y=4$ and the plane $z=0$.

Sol: Given, $\vec{V} = x^3\vec{i} - x^2y\vec{j}$

By Stokes theorem, $\oint_C \vec{V} \cdot d\vec{r} = \iint_S \text{curl } \vec{V} \cdot \vec{n} \, ds$

$$\oint_C (x^3\vec{i} - x^2y\vec{j}) (dx\vec{i} + dy\vec{j}) = \iint_S \text{curl } \vec{V} \cdot \vec{n} \, ds$$

$$\text{R.H.S} = \iint_S \text{curl } \vec{V} \cdot \vec{n} \, ds$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & -x^2y & 0 \end{vmatrix}$$

$$= \vec{i} \left(0 - \frac{\partial}{\partial z} (-x^2y) \right) - \vec{j} \left(0 - \frac{\partial}{\partial z} (x^3) \right) + \vec{k} \left(\frac{\partial}{\partial x} (-x^2y) - \frac{\partial}{\partial y} (x^3) \right)$$

$$= 0 - 0 + (-2xy - 0)\vec{k} = -2xy\vec{k}$$

$\vec{n}'$ is unit normal to the surface $z=0$ is $\vec{n} = \vec{k}$

$$\iint_S \text{curl } \vec{V} \cdot \vec{n} \, ds = \iint_S \text{curl } \vec{V} \cdot \vec{n} \, dx \, dy \cdot \frac{1}{|\vec{n} \cdot \vec{k}|}$$

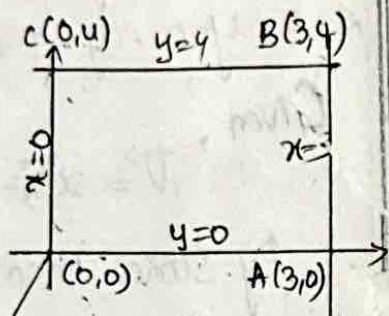
$$\iint (-2xyx) \cdot \vec{r} \, dx \, dy = \iint -2xy \, dx \, dy$$

$$-\int_0^3 \int_0^4 2xy \, dx \, dy = -\int_0^3 2x \left(\frac{y^2}{2} \right)_0^4 \, dx = -\int_0^3 x(16-0) \, dx$$

$$= -\int_0^3 16x \, dx = -16 \left(\frac{x^2}{2} \right)_0^3 = -16 \left(\frac{9}{2} \right) = -72.$$

$$\text{L.H.S: } \oint_C \vec{v} \cdot d\vec{r} = \int_{OA} \vec{v} \cdot d\vec{r} + \int_{AB} \vec{v} \cdot d\vec{r}$$

$$+ \int_{BC} \vec{v} \cdot d\vec{r} + \int_{CO} \vec{v} \cdot d\vec{r}.$$



$$\int_C \vec{v} \cdot d\vec{r} = \int_C (x^3 \vec{i} - x^2 y \vec{j}) (dx \vec{i} + dy \vec{j})$$

$$= \int x^3 dx - x^2 y dy$$

1* Along OA:- $y=0$ $x: (0 \rightarrow 3)$
 $dy=0$

$$\int_{OA} x^3 dx - x^2 y dy = \int_0^3 (x^3) dx - 0 = \left(\frac{x^4}{4} \right)_0^3 = \frac{81}{4}.$$

2* Along AB:- $x=3$ $y: (0 \rightarrow 4)$
 $dx=0$

$$\int_{AB} x^3 dx - x^2 y dy = \int_0^4 -x^2 y dy = -9 \left[\frac{y^2}{2} \right]_0^4$$

$$= \frac{-9 \times 16}{2} = -72.$$

3. Along BC: $y=4$ $x:(3 \rightarrow 0)$
 $dy=0$

$$\int_{BC} x^3 dx - x^2 y dy = \int_3^0 x^3 dx = \left(\frac{x^4}{4} \right)_3^0 = -\frac{81}{4}$$

4. Along CO: $x=0$ $y:(4 \rightarrow 0)$
 $dx=0$

$$\int_{CO} x^3 dx - x^2 y dy = \int_4^0 0 = 0$$

$$L.H.S = \int x^3 dx - x^2 y dy = \frac{81}{4} - 72 - \frac{81}{4} + 0 = -72$$

$$\therefore L.H.S = R.H.S$$

Handwritten text at the bottom of the page, likely a signature or date, is partially obscured and difficult to read.

Apply Green's theorem to Evaluate $\oint_C (y^2 - 2xy) dx + (x^2 + y^2) dy$
where 'C' is the closed surface formed by $y = x^2$ and $x = y^2$

By Green's theorem we have,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

$$\text{i.e. } \oint_C (y^2 - 2xy) dx + (x^2 + y^2) dy$$

$$\text{Here } M = y^2 - 2xy, \quad N = x^2 + y^2$$

$$\frac{\partial M}{\partial y} = 2y - 2x, \quad \frac{\partial N}{\partial x} = 2x$$

$$\text{i.e. } \iint_R (2x - 2y + 2x) dx dy$$

$$= 2 \int_{x=0}^1 \int_{y=x^2}^{\sqrt{x}} (2x - y) dy dx$$

$$= 2 \int_{x=0}^1 \left(2xy - \frac{y^2}{2} \right)_{x^2}^{\sqrt{x}} dx$$

$$= 2 \int_0^1 \left[\left(2x^{3/2} - \frac{x}{2} \right) - \left(2x^3 - \frac{x^4}{2} \right) \right] dx$$

$$= 2 \left[\frac{2x^{5/2}}{5/2} - \frac{x^2}{4} - \frac{2x^4}{4} + \frac{x^5}{10} \right]_0^1$$

$$= 2 \left[\frac{4}{5} - \frac{1}{4} - \frac{1}{2} + \frac{1}{10} \right]$$

$$= \frac{3}{10}$$

$$(1) \int_0^a \int_0^a xy \, dx \, dy \cdot \underline{\text{Ans}} \frac{a^4}{4}$$

$$(2) \int_1^a \int_1^b \frac{dx \, dy}{xy}$$

$$\int_{y=1}^b \int_{x=1}^a [\log x] \, dx = \log b [\log x]_1^a$$

$$= \log b \log a$$

$$= \log ab$$

$$(3) \int_0^1 \int_0^1 \frac{1}{\sqrt{(1-x)(1-y)}} \, dx \, dy$$

$$\int_{y=0}^1 \left[\sin^{-1} x \right]_0^1 dy = \sin^{-1} \left[\frac{\pi}{2} \right] = \frac{\pi}{4}$$

$$(4) \int_1^2 \int_0^2 (x^3 + y^3) \, dx \, dy = \frac{23}{2}$$

$$(5) \int_1^2 \int_0^3 (x^2 y + xy^2) \, dx \, dy = 24$$

$$(6) \int_0^1 \int_0^{x^2} e^{y/x} \, dx \, dy$$

$$\int_0^1 \left[\frac{e^{y/x}}{1/x} \right]_0^{x^2} dx$$

$$\int_0^1 x e^x = \int_0^1 [x e^x - x]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$$

$$(7) \int_0^1 \int_0^{1-x} e^{2x+3y} \, dx \, dy$$

$$\int_0^1 \left[\frac{e^{2x+3y}}{3} \right]_0^{1-x} dx = \frac{1}{3} \int_0^1 \left[e^{\frac{2x+3(1-x)}{3}} - e^{2x} \right] dx$$

$$= \frac{1}{3} \int_0^1 \left[e^{3-x} - e^{2x} \right] dx = \frac{1}{3} \left[\frac{e^{3-x}}{-1} - e^{2x} \right]_0^1 = \frac{1}{6} [2e^3 - 3e^2]$$

$$(8) \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{1+x^2+y^2} dy dx$$

$$\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1-x^2}} dy$$

$$\int_0^1 \frac{1}{\sqrt{1+x^2}} [\tan^{-1} 1 - \tan^{-1} 0] dx$$

$$\frac{\pi}{4} [\sinh]_0^1 = \frac{\pi}{4} \cdot \frac{\pi}{2} = \frac{\pi}{4} [\sinh 1]$$

$$(9) \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2-y^2}} dy dx \quad \int \sqrt{a^2-x^2} dx = \left[\frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]$$

$$= \int_0^a \left[\frac{y}{2} \sqrt{a^2-x^2-y^2} + \frac{(a^2-x^2)}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \left[\frac{\sqrt{a^2-x^2}}{2} \sqrt{a^2-x^2-y^2} + \frac{(a^2-x^2)}{2} \sin^{-1} \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} \right] dx$$

$$= \int_0^a \frac{a^2-x^2}{2} \frac{\pi}{2} dx$$

$$= \frac{\pi}{4} \left[a^2 x - \frac{x^3}{3} \right]_0^a = \frac{\pi a^3}{6}$$

$$(10) \int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx dy dz$$

$$\int_0^1 \int_0^1 \int_0^1 e^x \cdot e^{y+z} = (e-1)^3$$

$$= (e-1)^3$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xy^2 dx dy dz = \frac{1}{48}$$

$$(12) \int_1^3 \int_{1/n}^1 \int_0^{\sqrt{xy}} xyz \, dz \, dy \, dx$$

(2)

$$\frac{1}{2} \int_1^3 \int_{1/n}^1 xy^2 \, dy \, dx$$

$$\frac{1}{6} \int_1^3 n^2 \left(\frac{1}{3} - \frac{1}{n^3} \right) \, dx$$

$$\frac{1}{6} \int_1^3 \left(x^2 - \frac{1}{n} \right) \, dx = \frac{1}{6} \left[\frac{x^3}{3} - \log x \right]_1^3 = \frac{1}{6} \left(\frac{27}{3} - \log 3 - \frac{1}{3} + 1 \right) \\ = \frac{1}{6} \left(\frac{26}{3} - \log 3 \right)$$

$$(3) \int_{-1}^1 \int_0^x \int_{x-2}^{x+2} (x+y+z) \, dy \, dz \, dx = 0$$

$$(14) \int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz \, dx \, dy$$

$$\int_{y=1}^e \int_{x=1}^{\log y} \int_{z=1}^{e^x} \log z \, dz \, dx \, dy$$

$$\int_{y=1}^e \int_{x=1}^{\log y} \left[z \log z - z \right]_1^{e^x} \, dx \, dy$$

$$\int_{y=1}^e \int_{x=1}^{\log y} \left[e^x \log e^x - e^x \right]_1^{e^x} \, dx \, dy$$

$$\int_{y=1}^e \int_{x=1}^{\log y} \left[e^x (\log e^x - 1) - 1 (\log 1 - 1) \right] \, dx \, dy$$

$$\int_{y=1}^e \int_{x=1}^{\log y} \left[e^x (x - 1) + 1 \right] \, dx \, dy$$

$$\int_1^e \int_1^{\log y} \left[x e^x - e^x + 1 \right] \, dx \, dy = \int_1^e \left[x e^x - e^x + 1 \right]_1^{\log y} \, dy$$

$$\int_1^e \left[\log y (e^{\log y} - e^1 + 1) \right] \, dy$$