UNIT-IV Multivariable Calculus

MULTIPLE INTEGRALS, DOUBLE AND TRIPLE INTEGRALS

Evaluate $\iint xy(x+y)\,dx\,dy$ over the area between $y=x^2$ and $y=x$.

$y=x^2,\ y=x$

$x^2=x$

$x^2-x=0$

$x(x-1)=0$

$x=0,\ x=1$

$$\iint xy(x+y)\,dx\,dy$$

$$\int_{x=0}^{x=1}\int_{y=x^2}^{y=x} xy(x+y)\,dx\,dy$$

$$=\int_{x=0}^{1}\int_{x^2}^{x}(x^2y+xy^2)\,dx\,dy$$

$$=\int_{x=0}^{1}\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{x^2}^{x}dx$$

$$=\int_{x=0}^{1}\left[\frac{x^4}{2}+\frac{x^4}{3}-\frac{x^6}{2}-\frac{x^7}{3}\right]dx$$

$$=\left[\frac{x^5}{10}+\frac{x^5}{15}-\frac{x^7}{14}-\frac{x^8}{24}\right]_{x=0}^{x=1}$$

$$=\frac{1}{10}+\frac{1}{15}-\frac{1}{14}-\frac{1}{24}=\frac{84+56-60-35}{840}=\frac{45}{840}=\frac{3}{56}$$

EVALUATE $\iint_D (x^2+y^2)\,dx\,dy$, where "D" is the region bounded by $y=x^2,\ x=2,\ x=1$.

$$\iint(x^2+y^2)\,dx\,dy$$

$$\int_{x=1}^{2}\int_{y=0}^{x^2}(x^2+y^2)\,dx\,dy$$

$$=\int_{x=1}^{2}\left(x^2y+\frac{y^3}{3}\right)_0^{x^2}dx=\int_{x=1}^{2}\left(x^4+\frac{x^6}{3}\right)dx$$

$$=\left(\frac{x^5}{5}+\frac{x^7}{21}\right)_1^2=\frac{32}{5}+\frac{128}{21}-\frac{1}{5}-\frac{1}{21}$$

$$=\frac{672+640-21-5}{105}=\frac{1286}{105}$$

Q3: INTEGRATE $\int_0^a\int_0^b xy(x-y)\,dx\,dy=\int_0^b\int_0^a xy(x-y)\,dx\,dy$.

Sol: L.H.S :- $\int_{x=0}^{a}\int_{y=0}^{b} xy(x-y)\,dx\,dy$

$$=\int_{x=0}^{a}\int_{y=0}^{b}(x^2y-xy^2)\,dx\,dy$$

$$=\int_{x=0}^{a}\left(\frac{x^2y^2}{2}-\frac{xy^3}{3}\right)_0^b dx=\int_0^a\left(\frac{x^2b^2}{2}-\frac{xb^3}{3}\right)dx$$

$$=\left(\frac{b^2x^3}{6}-\frac{b^3x^2}{6}\right)_0^a=\frac{a^3b^2}{6}-\frac{a^2b^3}{6}=\frac{a^2b^2(a-b)}{6}\rightarrow(1)$$

R.H.S:-

$$\int_{y=0}^{b}\int_{x=0}^{a}(x^2y - xy^2)\,dx\,dy$$

$$= \int_{y=0}^{b}\left(\frac{x^3y}{3} - \frac{x^2y^2}{2}\right)_0^a dy = \int_{y=0}^{b}\left(\frac{a^3y}{3} - \frac{a^2y^2}{2}\right) dy$$

$$= \left(\frac{a^3y^2}{6} - \frac{a^2y^3}{6}\right)_0^b = \frac{a^3b^2 - a^2b^3}{6} = \frac{a^2b^2(a-b)}{6} \rightarrow (2)$$

$$\therefore (1) = (2)$$

$$\therefore \int_0^a\int_0^b xy(x-y)\,dx\,dy = \int_0^b\int_0^a xy(x-y)\,dx\,dy.$$

Q4: $\iint (x^2+y^2)\,dx\,dy$ in +ve Quadrant $x+y\leq 1$.

Sol:

$x+y=1$
$x=0,\ x=1-y$
$y=0,\ y=1$
or
$y=0,\ y=1-x$
$x=0,\ x=1$

$$\iint (x^2+y^2)\,dx\,dy$$

$$\int_{x=0}^{1}\int_{y=0}^{1-x}(x^2+y^2)\,dx\,dy = \int_{x=0}^{1}\left(x^2y + \frac{y^3}{3}\right)_0^{1-x} dx$$

$$= \int_0^1\left[x^2(1-x) + \frac{(1-x)^3}{3}\right]dx$$

$$= \int_0^1\left[x^2 - x^3 + \frac{1}{3}\left[1-3x+3x^2-x^3\right]\right]dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 - \left(\frac{x^4}{4}\right)_0^1 + \left(\frac{x}{3}\right)_0^1 - \left(\frac{x^2}{2}\right)_0^1 + \left(\frac{x^3}{3}\right)_0^1 - \left(\frac{x^4}{12}\right)_0^1$$

$$= \frac{1}{3} - \frac{1}{4} + \frac{1}{3} - \frac{1}{2} + \frac{1}{3} - \frac{1}{12} = \frac{2}{12} = \frac{1}{6}.$$

Q5. CHANGE THE ORDER OF INTEGRATION AND Hence Evaluate

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\, dx\, dy$$

Sol:

$$= \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{1-y^2}} y^2\, dx\, dy$$

$x = 0,\ x = 1$

$y = 0,\ y = \sqrt{1-x^2}$

$y^2 = 1 - x^2 \Rightarrow x^2 + y^2 = 1$

change of order of Integration

$y = 0,\ y = 1$

$x^2 = 1 - y^2,\ x = 0$

$x = \sqrt{1-y^2}$

$$= \int_0^1 y^2 (x)_0^{\sqrt{1-y^2}}\, dy$$

$$= \int_0^1 y^2 (\sqrt{1-y^2})\, dy$$

Limits:

y	0	1
$\sin\theta$	0	$\pi/2$

Let $y = \sin\theta$

$\frac{dy}{d\theta} = \cos\theta \Rightarrow dy = \cos\theta\, d\theta$

$$\Rightarrow \int_0^{\pi/2} \sin^2\theta . \cos^2\theta\, d\theta = \frac{1}{4} \cdot \frac{1}{2} \cdot \pi/2 = \pi/16.$$

Q6: CHANGE OF ORDER OF INTEGRATION AND Hence Evaluate

$$\int_0^b \int_0^{\frac{a\sqrt{b^2-y^2}}{b}} xy\, dx\, dy$$

$$\int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}\sqrt{a^2-x^2}} xy\,dx\,dy$$

$$= \int_{x=0}^{a} x\left(\frac{y^2}{2}\right)_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \frac{1}{2}\int_{x=0}^{a} x\left(\frac{b^2}{a^2}(a^2-x^2)\right) dx$$

$$= \frac{b^2}{2a^2}\int_{x=0}^{a} (a^2x - x^3)\,dx$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right)$$

$y=0,\ y=b$

$x=0,\ x=\frac{a\sqrt{b^2-y^2}}{b}$

$x^2 = \frac{a^2}{b^2}(b^2-y^2)$

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Change of order of integration,

$x=0,\ x=a$

$\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$y^2 = \frac{b^2(a^2-x^2)}{a^2}$

$y = \frac{b}{a}\sqrt{a^2-x^2}$.

$y=0$.

$$= \frac{b^2}{2a^2}\times\frac{2a^4-a^4}{4} = \frac{b^2a^4}{8a^2} = \frac{b^2a^2}{8}.$$

Evaluate $\int_0^2\int_1^2 (x^3+y^3)\,dx\,dy$.

$$\int_{x=0}^{2}\int_{y=1}^{2} (x^3+y^3)\,dx\,dy$$

$$= \int_{x=0}^{2}\left[x^3y + \frac{y^4}{4}\right]_1^2 dx$$

$$= \int_0^2 \left[x^3 + \left(\frac{16-1}{4}\right)\right] dx$$

$$= \int_0^2 x^3 dx + \frac{15}{4} dx = \left(\frac{x^4}{4}\right)_0^2 + \frac{15}{4}(x)_0^2$$

$$= \frac{16}{4} - 0 + \frac{15}{4_2} \times 2 = 4 + \frac{15}{2} = \frac{23}{2}.$$

Q8: Evaluate $\int_0^3 \int_1^2 (x^2y + xy^2)\, dx\, dy$.

Sol: $$\int_{x=0}^{x=3} \int_{y=1}^{y=2} (x^2y + xy^2)\, dx\, dy = \int_{x=0}^{3} \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_1^2 dx$$

$$= \int_0^3 \left[\left(\frac{4x^2}{2} + \frac{8x}{3}\right) - \left(\frac{x^2}{2} + \frac{x}{3}\right)\right] dx$$

$$= \int_0^3 2x^2 dx + \frac{8}{3}\int_0^3 x\, dx - \frac{1}{2}\int_0^3 x^2 dx - \frac{1}{3}\int_0^3 x\, dx$$

$$= 2\left(\frac{x^3}{3}\right)_0^3 + \frac{8}{3}\left(\frac{x^2}{2}\right)_0^3 - \frac{1}{2}\left(\frac{x^3}{3}\right)_0^3 - \frac{1}{3}\left(\frac{x^2}{2}\right)_0^3$$

$$= 2\left(\frac{\cancel{27}^{9}}{\cancel{3}}\right) + \frac{\cancel{8}^{4}}{3}\left(\frac{\cancel{9}^{3}}{\cancel{2}}\right) - \frac{1}{2}\left(\frac{\cancel{27}^{9}}{\cancel{3}}\right) - \frac{1}{3}\left(\frac{9}{2}\right)$$

$$= 18 + 12 - \frac{9}{2} - \frac{3}{2} = 30 - \frac{12}{2} = 30 - 6$$

$$= 24.$$

Triple Integrals

If the region 'v' bounded by sphere surfaces $x=x_1$, $x=x_2$, $y=y_1$, $y=y_2$, $z=z_1$, $z=z_2$ then

$$\iiint_V (x,y,z)\,dx\,dy\,dz.$$

$$= \int_{x=x_1}^{x=x_2}\int_{y=y_1}^{y=y_2}\int_{z=z_1}^{z=z_2} (x,y,z)\,dx\,dy\,dz.$$

Q1: Evaluate $\int_0^{2a}\int_0^{x}\int_y^{x} xyz\,dx\,dy\,dz$

Sol: $= \int_0^{2a}\int_0^{x} xy\left[\frac{z^2}{2}\right]_y^x dx\,dy = \frac{1}{2}\int_0^{2a}\int_0^{x} xy(x^2-y^2)\,dx\,dy$

$$= \frac{1}{2}\int_0^{2a}\int_0^{x}(x^3y - xy^3)\,dx\,dy = \frac{1}{2}\int_0^{2a}\left(\frac{x^3y^2}{2} - \frac{xy^4}{4}\right)_0^x dx$$

$$= \frac{1}{2}\int_0^{2a}\left(\frac{x^3\cdot x^2}{2} - \frac{x\cdot x^4}{4}\right)dx$$

$$= \frac{1}{2}\int_0^{2a}\left(\frac{x^5}{2} - \frac{x^5}{4}\right)dx = \frac{1}{8}\int_0^{2a}(2x^5 - x^5)\,dx$$

$$= \frac{1}{8}\int_0^{2a} x^5\,dx = \frac{1}{8}\left(\frac{x^6}{6}\right)_0^{2a} = \frac{1}{8}\,\frac{(2a)^6}{6}$$

$$= \frac{1}{48} \times 32 \times 2 \times a^6 = \frac{4a^6}{3}.$$

Q2: $\int_0^1\int_0^x\int_0^{x+y}(x+y+z)\,dx\,dy\,dz.$

Sol: $\int_0^1\int_0^x\left(xz+yz+\frac{z^2}{2}\right)_0^{x+y}dx\,dy$

$=\int_0^1\int_0^x\left(x(x+y)+y(x+y)+\frac{(x+y)^2}{2}\right)dx\,dy$

$=\int_0^1\int_0^x\left(x^2+xy+xy+y^2+x^2+y^2+\frac{2xy}{2}\right)dx\,dy$

$=\int_0^1\int_0^x\left(2x^2+2xy+\frac{\cancel{2}xy}{\cancel{2}}+2y^2\right)dx\,dy$

$=\int_0^1\int_0^x\left(2x^2+3xy+2y^2\right)dx\,dy$

$=\int_0^1\left(2x^2y+\frac{3xy^2}{2}+\frac{2y^3}{3}\right)_0^x dx$

$=\int_0^1\left(2x^3+\frac{3x^3}{2}+\frac{2x^3}{3}\right)dx=\int_0^1\left(2x^3+\frac{3x^3}{2}+\frac{2x^3}{3}\right)dx$

$=\int_0^1\frac{25x^3}{6}dx=\frac{25}{24}\left(x^4\right)_0^1=\frac{25}{24}.$

$$\int_0^2\int_1^z\int_0^{yz} xyz\,dx\,dy\,dz.$$

$$\int_0^2\int_1^z\int_0^{yz} xyz\,dx\,dy\,dz = \int_0^2\int_1^z yz\left(\frac{x^2}{2}\right)_0^{yz} dy\,dz$$

$$\int_0^2\int_1^z yz\left(\frac{y^2z^2}{2}\right)dy\,dz = \int_0^2\int_1^z\left(\frac{y^3z^3}{2}\right)dy\,dz$$

$$= \frac{1}{2}\int_0^2 z^3\left(\frac{y^4}{4}\right)_1^z dz = \frac{1}{2}\int_0^2 z^3\left(\frac{z^4}{4}-\frac{1}{4}\right)dz$$

$$= \frac{1}{2}\int_0^2\left(\frac{z^7}{4}-\frac{z^3}{4}\right)dz = \frac{1}{2}\times\frac{1}{4}\int_0 (z^7-z^3)\,dz.$$

$$= \frac{1}{8}\left[\frac{z^8}{8}-\frac{z^4}{4}\right]_0^2 = \frac{1}{8}\left[\frac{2^8}{8}-\frac{2^4}{4}-0\right]$$

$$= \frac{1}{8}\left[\frac{\cancel{256}^{\,32}}{\cancel{8}_{1}}-\frac{\cancel{16}^{\,4}}{\cancel{4}_{1}}\right] = \frac{1}{\cancel{8}_{4}}\left(\cancel{28}^{\,14}\right) = \frac{14}{4} = \frac{7}{2}.$$

xy^2

(1) $\int_0^a \int_0^a xy\,dx\,dy$. Ans $\frac{a^4}{4}$

(2) $\int_1^a \int_1^b \frac{dx\,dy}{xy}$

$\int_{y=1}^{y=b} [\log x]\,dy = \log b\,[\log y]_1^a$

$= \log b \log a$

$= \log ab$

(3) $\int_0^1 \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy$

$\int_{y=0}^{y=1} [\sin^{-1}x]_0^a\,dy = \sin^{-1}\pi/2\,[\sin^{-1}x] = \frac{\pi^2}{4}$

(4) $\int_1^2 \int_0^2 (x^3+y^3)\,dx\,dy = \frac{23}{2}$

(5) $\int_1^2 \int_0^3 (x^2y+xy^2)\,dx\,dy = 24$

(6) $\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy$

$\int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$

$\int_0^1 xe^x = \int_0^1 [xe^x - x]_0^1 = -\frac{1}{2}+1 = \frac{1}{2}$

(7) $\int_0^1 \int_0^{1-x} e^{2x+3y}\,dx\,dy$

$\int_0^1 \left[\frac{e^{2x+3y}}{3}\right]_0^{1-x} dx = \frac{1}{3}\int_0^1 \left[e^{2x+3(1-x)} - e^{2x}\right] dy$

$= \frac{1}{3}\int_0^1 [e^{3-x} - e^{2x}]\,dx = \frac{1}{3}\left[\frac{e^{3-x}}{-1} - e^{2x}\right]_0^1 = -\frac{1}{6}[2e^3 - 3e$

$\left[\frac{1}{\sqrt{a}} \tan \frac{\pi}{4}\right]$

(8) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

$\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$

$\int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \left[\tan^{-1} 1 - \tan^{-1} 0\right] dx$

$\frac{\pi}{4}\left[\sinh^{-1} x\right]_0^1 = \cancel{\frac{\pi}{4}}\pi/2 = \frac{\pi}{4}\left[\sinh^{-1}(1)\right]$

(9) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$ $\quad \int \sqrt{a^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$= \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{(a^2-x^2)}{2} \sin^{-1} \frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$

$= \int_0^a \left[\underbrace{\frac{\sqrt{a^2-x^2}}{2} \sqrt{a^2-x^2-(a^2-x^2)}}_{(0)} + \frac{(a^2-x^2)}{2} \sin^{-1} \frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right] dx$

$= \int_0^a \frac{a^2-x^2}{2}\, \pi/2\, dx$

$= \frac{\pi}{4}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{\pi a^3}{6}$

(10) $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\, dy\, dz$

$\int_0^1 \int_0^1 \int_0^1 e^x \cdot e^{y+z} = (e-1)^3$

$= (e^1-1)^3$

(11) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz = \frac{1}{48}$

(12) $\int_1^3 \int_{1/x}^1 \int_0^{\sqrt{xy}} xyz\, dx\, dy\, dz$

$\frac{1}{2}\int_1^3 \int_{1/x}^1 xy^2\, dy\, dz$

$\frac{1}{6}\int_1^3 x^2 (\cancel{} (1 - 1/x^3)$

$\frac{1}{6}\int_1^3 [x^2 - \frac{1}{x}]\, dx = \frac{1}{6}[\frac{x^3}{3} - \log x]_1^3 = 1/6 [\frac{27}{3} - \log 3 - \frac{1}{3} + \log 1]$

$= \frac{1}{6}[\frac{26}{3} - \log 3]$

(13) $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz = 0$

(14) $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z\, dz\, dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{x=\log y} \int_{z=1}^{z=e^x} \cancel{z\,[\log z - 1]}\ \log z\, dx\, dy\, dz$

$\int_{y=1}^{e} \int_{x=1}^{x=\log y} [z \log z - z]_1^{e^x}\, dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{x=\log y} [e^x \log e^x - e^x]_1^{e^x}\, dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{x=y} \cancel{e^x [\log e^x - 1]\, dx} \ [e^x [\log e^x - 1] - 1 [\log 1 - 1]]\, dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{\log y} [e^x [x - 1] + 1]\, dx\, dy$

$\int_1^e \int_1^{\log y} [x e^x - e^x + 1)\, dx\, dy = \int [x e^x - e^x \cancel{+} e^x + 1]_1^{\log y}\, dy$

$\int \cancel{[\log y\, e^{y} \ldots]}\, dy$

$$\int_1^e \left[\log y \cdot (\cancel{\log y} - e^{\log y} - 2e^{\log y} + \log y) - \{e - 2e + 1\}\right] dy$$

$$\int_1^e \left[y \log y - 2y + \log y + e - 1\right] dy$$

$$\int_1^e \cancel{\frac{y^2}{2}\log y - \frac{y^2}{4}} \left[\log y\,(y+1) - 2y + e - 1\right] dy$$

$$\log y \left[\frac{y^2}{2} + 1\right] - \int \frac{1}{y}\left(\frac{y^2}{2} + y\right) - y^2 + ey - y\Big]_1^e$$

$$\left[\log y \left[\frac{y^2}{2} + y\right] - \frac{y^2}{4} - y - y^2 + ey - y\right]_1^e$$

$$\left[\log e \left[\frac{e^2}{2} + y\right] - \frac{e^2}{4} - e - \cancel{e^2} + \cancel{e^2} - e\right]$$

$$- \left[\cancel{0} - 1/4 \cancel{1} - 1 + e - 1\right]$$

$$1\left[\frac{e^2}{2} + e\right] - \frac{e^2}{4} - 2e + \frac{13}{4} - e$$

$$= \frac{2e^2 + 4e - e^2 - 8e + 13 - \cancel{4e}}{4}$$

$$= \frac{e^2 - 8e + 13}{4} \quad \text{Ans}$$

Change of order of Integration

(1). Find the Double Integrals of the following by using change of order of Integration

(1) $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 \, dy \, dx =$

(2) $\int_0^a \int_{y = x/a}^{y = \sqrt{x}/a} (x^2 + y^2) \, dx \, dy$

$y = \frac{\sqrt{x}}{a} \Rightarrow x = ay^2$

$y = x/a \Rightarrow x = ay$

~~y~~ $ay = a$

$x = 0, \Rightarrow y = 0$

$x = a \Rightarrow y = 1$

Sol: $\int_{x=0}^{x=a} \int_{y = x/a}^{y = \frac{\sqrt{x}}{a}} (x^2 + y^2) \, dx$

③

$$\boxed{\begin{aligned}&\int_{y=0}^{y=1}\int_{x=ay}^{x=ay^2}(x^2+y^2)\,dx\,dy\\&\int_{y=0}^{y=1}\left[\frac{x^3}{3}y+\frac{y^3}{3}x\right]_{ay}^{ay^2}dx\\&\int_{y=0}^{y=1}\left[\frac{a^3y^6}{3}+y^2ay^2-\right.\end{aligned}}$$

$$\int_{y=0}^{y=1}\int_{x=ay}^{ay^2}(x^2+y^2)\,dx\,dy$$

$$\int_{y=0}^{1}\left[\frac{x^3}{3}+y^2x\right]_{ay}^{ay^2}dx$$

$$\int_{y=0}^{1}\left[\frac{a^3y^6}{3}+y^2ay^2-\frac{a^3y^3}{3}-y^2ay\right]dy$$

$$\left[\frac{a^3y^7}{21}+\frac{ay^5}{5}-\frac{a^3y^4}{12}-\frac{y^4}{4}a\right]_0^1$$

$$\frac{a^3}{21}-\frac{a^3}{12}+\frac{a}{5}-\frac{a}{4}$$

$$\frac{4a^3-7a^3}{84}+\frac{4a-5a}{20}$$

$$\frac{-3a^3}{84}-\frac{a}{20}=\underline{-\left(\frac{a^3}{28}+\frac{a}{20}\right)}\ \underline{Ans}$$

③ $\int_0^a\int_y^a\frac{x}{x^2+y^2}\,dx\,dy$ $\qquad x=y,\ x=a$ & $y=0,\ y=a$

$$\int_{y=0}^{y=a}\int_{x=y}^{x=a}\qquad\Longrightarrow\quad \begin{matrix}x=0, & x=a\\ y=a & y=0\end{matrix}$$

$$\int_{y=0}^{y=a}\int_{x=0}^{x=a}\frac{x}{x^2+y^2}\,dx\,dy=\int_{x=0}^{x=a}x\,dx\left[\frac{1}{x}\tan^{-1}\frac{y}{x}\right]_0^a dx$$

$$= \int_{x=0}^{x=a} [\tan^{-1} 1 - \tan^{-1} 0]\, dx$$

$$= \frac{\pi}{4} \int_{x=0}^{x=a} dx = \frac{\pi a}{4}$$

Directional Derivative :-

$$D.D = \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$$

$$D.D = \nabla f \cdot \frac{\nabla \phi}{|\nabla \phi|}$$

$$D.D = \nabla f \cdot \frac{\vec{PQ}}{|PQ|}$$

~~Greatest~~

Greatest or Maximum value :- the greatest & maximum value of $f(x,y,z)$ at pt P is given by $|\nabla f|$.

① Find the D.D of the function $xyz^2 + xz$ at $(1,1,1)$ in the direction of normal to the surface $3xy^2 + y = z$ at $(0,1,1)$

$$\nabla f_{(1,1,1)} = 2i + j + 3k$$

$$\nabla \phi_{(0,1,1)} = 3i + j - k$$

$$D.D = \nabla f \cdot \frac{(\nabla \phi)}{|\nabla \phi|} = \frac{4}{\sqrt{11}}$$

② ~~Q~~ Find the greatest or maximum value of the function x^2yz^3 at a pt $(2,1,-1)$

$$|\nabla f| = 4\sqrt{11}$$