SCALAR AND VECTOR FIELD

SCALAR FIELD:

A Point function that assigns a Scalar to every point in Some region in the plane or in space is called a Scalar field.

Eg: Temperature distribution throughout the body.

VECTOR FIELD:

A point function that assigns a Vector to every point in Some region in plane or in space is called a Vector field.

Eg: Electric field and magnetic field

VECTOR DIFFERENTIAL OPERATOR:

The Operator "∇" (Dell) is defined as $\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$

"∇" is called a Vector differential operators, it involves the unit vector i, j, k and also the partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$.

GRADIENT:

Let $\phi(x, y, z)$ be differential at each point in a certain region, then Gradient of ϕ, it is written as $\nabla\phi$.

i.e $\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$

$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

$\nabla\phi = \text{grad}\,\phi$

UNIT NORMAL TO SURFACE: Let $\phi(x,y,z) = c$, then

$n = \frac{\nabla\phi}{|\nabla\phi|}$ is unit Normal to surface at a point

$p(x,y,z)$.

ANGLE BETWEEN TWO SURFACES:

Let $\phi_1(x,y,z) = c_1$ and $\phi_2(x,y,z) = c_2$ be the two surfaces at a common point p, the angle between the two surfaces is the angle between the two normals.

$$\cos\theta = \frac{(\nabla\phi_1)(\nabla\phi_2)}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\theta = \cos^{-1}\left[\frac{(\nabla\phi_1)(\nabla\phi_2)}{|\nabla\phi_1||\nabla\phi_2|}\right]$$

DIRECTIONAL DERIVATIVE:

The directional derivative of $F(x,y,z)$ in direction of $\vec{a}$ vector is given by $\nabla F \frac{(\vec{a})}{|\vec{a}|}$.

Also the directional derivative of F at a point 'p' is maximum in the direction of Normal at "p" and its maximum value $= |\nabla\phi|$.

Q1: Find Unit Normal to Surface $x^3+y^3+3xyz=3$ at the point $(1,2,-1)$.

Sol: Let $\phi(x,y,z) = x^3+y^3+3xyz-3$, $P(1,2,-1)$

Unit Normal to the surface $\phi(x,y,z)$, $n = \frac{\nabla\phi}{|\nabla\phi|}$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^3+y^3+3xyz-3)$$

$$= i(3x^2+3yz) + j(3y^2+3xz) + k(3xy)$$

$$\nabla\phi_{(1,2,-1)} = i(3-6) + j(12-3) + k(6)$$

$$= -3i + 9j + 6k$$

$$|\nabla\phi| = \sqrt{9+81+36} = 3\sqrt{14} = \sqrt{126} = \sqrt{9\times 14}$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-3i+9j+6k}{3\sqrt{14}} = \frac{-i+3j+2k}{\sqrt{14}}$$

Q2: If $\phi(x,y,z) = 3x^2y - y^3z^3$ at a point $P(1,-2,-1)$. Find grad ϕ.

Sol:
$$\phi(x,y,z) = 3x^2y - y^3z^3$$

$$\text{grad}\,\phi = \nabla\phi$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(3x^2y - y^3z^3)$$

$$= i(6xy) + j(3x^2-3y^2z^3) + k(-3z^2y^3)$$

$$\text{grad}\,\phi(1,-2,-1) = (6)(1)(-2)i + j(3-3(4)(-1)) + 3(-1)^2 k(8)$$

$$\text{grad}\,\phi_{(1,-2,-1)} = -12i + 15j + 24k$$

Q4: If θ is the acute angle b/w the surfaces $xy^2z = 3x + z^2$ and $3x^2 + 2z - y^2 = 1$ at pt $(1,-2,1)$. S.T $\cos\theta = 3/7\sqrt{6}$.

Sol: $\phi_1(x,y,z) = xy^2z - 3x - z^2$, $\phi_2(x,y,z) = 3x^2 + 2z - y^2 - 1$

at a pt $(1,-2,1)$

The angle between surfaces ϕ_1 & ϕ_2 is

$$\cos\theta = \frac{(\nabla\phi_1)(\nabla\phi_2)}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\nabla\phi_1 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy^2z - 3x - z^2)$$

$$= i(zy^2 - 3) + j(2xyz) + k(xy^2 - 2z)$$

$$\nabla\phi_1(1,-2,1) = i(4-3) + j(-4) + k(4-2)$$

$$= i - 4j + 2k.$$

$$|\nabla\phi_1| = \sqrt{1+16+4} = \sqrt{21}.$$

$$\nabla\phi_2 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(3x^2 - y^2 + 2z - 1)$$

$$= i(6x) + j(-2y) + k(2) = 6xi - 2yj + 2k$$

$$\nabla\phi_2(1,-2,1) = 6i + 4j + 2k$$

$$|\nabla\phi_2| = \sqrt{36+16+4} = \sqrt{56}$$

$$\cos\theta = \frac{(\nabla\phi_1)\cdot(\nabla\phi_2)}{|\nabla\phi_1||\nabla\phi_2|} = \frac{(i - 4j + 2k)(6i + 4j + 2k)}{\sqrt{21}\times\sqrt{56}}$$

Q3: Find the Angle b/w two surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at $(2,-1,2)$.

Sol: Let $\phi_1(x,y,z) = x^2+y^2+z^2-9$

$\phi_2(x,y,z) = z-x^2-y^2+3$

Angle b/w two surfaces is the Angle between the two normals i.e $\theta = \cos^{-1}\left[\frac{(\nabla\phi_1)(\nabla\phi_2)}{|\nabla\phi_1||\nabla\phi_2|}\right]$

$$\nabla\phi_1 = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= 2(xi+yj+zk)$$

$$\nabla\phi_{(2,-1,2)} = 2(2i-j+2k) = 4i-2j+4k.$$

$$|\nabla\phi_1| = \sqrt{16+4+16} = 6.$$

$$\nabla\phi_2 = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(z-x^2-y^2+3)$$

$$= -2xi-2yj+k$$

$$\nabla\phi_2(2,-1,2) = (-2)(2)i-(2)(-1)j+k = -4i+2j+k$$

$$|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21}.$$

$$\theta = \cos^{-1}\left[\frac{(4i-2j+4k).(-4i+2j+k)}{6\sqrt{21}}\right]$$

$$\theta = \cos^{-1}\left(\frac{-16-4+4}{6\sqrt{21}}\right) = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

$$\theta = 54.41^\circ.$$

4 Ans
Cont

$$= \frac{6-16+4}{\sqrt{3\times7\times8\times7}} = \frac{-6}{7\times2\sqrt{6}} = \frac{-3}{7\sqrt{6}}.$$

$$\cos\theta = 3/7\sqrt{6}.$$

Q5: In what direction $(3,1,-2)$ is director derivative of $\phi=x^2y^2z^4$ maximum. Find the magnitude of this maximum.

Sol: The directional derivative If N is the normal at the point $(3,1,-2)$ then $N = |\nabla\phi|_{(3,1,-2)}$.

$$\nabla\phi = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x^2y^2z^4)$$

$$= i(2xy^2z^4)+j(2x^2yz^4)+k(4x^2y^2z^3)$$

$$= i(2\times3\times1\times16)+j(2\times9\times16)+k(4\times9\times-8)$$

$$= 2\times3\times16(i+3j-3k)$$

$$\nabla\phi = 96(i+3j-3k), \quad |\nabla\phi| = 96\sqrt{1+9+9} = 96\sqrt{19}$$

$\therefore$ THE MAX. VALUE $= 96\sqrt{19}$.

Q6: Find the directional derivative of $f(x,y,z)=4e^{2x-y+z}$ at pt $(1,1,-1)$ in the direction towards the point $(-3,5,6)$.

Sol: Given,

$$\vec{a} = -3i+5j+6k$$

$$|\vec{a}| = \sqrt{9+25+36} = \sqrt{70}.$$

The Directional derivative of f in the direction of $\vec{a}$ is $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

$$\nabla f = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left(4e^{2x-y+z}\right)$$

$$= i\,4e^{2x-y+z}(2) + j\cdot 4e^{2x-y+z}(-1) + k\,4e^{2x-y+z}(1)$$

$$\nabla f_{(1,1,-1)} = 4e^{2-1-1}[2i - j + k]$$

$$\nabla f = 4(2i - j + k)$$

then Directional Derivative $= \nabla f \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= \frac{4(2i - j + k)(-3i + 5j + 6k)}{\sqrt{70}}$$

$$= \frac{4(-6-5+6)}{\sqrt{70}} = \frac{-20}{\sqrt{70}}.$$

Q7: Find Directional Derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,1)$ in direction $2i - j - 2k$.

Sol: $\vec{a} = 2i - j - 2k$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2yz + 4xz^2)$$

$$= i(2xyz + 4z^2) + j(x^2z) + k(x^2y + 8xz)$$

$$\nabla\phi_{(1,-2,1)} = i(2(1)(-2)(1) + 4(1)^2) + j(1\cdot 1) + k(1(-2)+8$$

$$\nabla\phi = j + 6k.$$

$$D.D = \nabla\phi\,\frac{\vec{a}}{|\vec{a}|} = \frac{(j+6k)(2i - j - 2k)}{3}$$

$$= \frac{-1-12}{3} = \frac{-13}{3}.$$

Q8: Find D.D of $\phi = xy^2 + yz^3$ at the point $(2,-1,1)$ in direction $i+2j+2k$.

Sol: $\vec{a} = i+2j+2k$

$|\vec{a}| = \sqrt{1+4+4} = 3$

$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xy^2+yz^3)$

$= y^2 i + j(2xy+z^3) + k(3z^2y)$

$\nabla\phi_{(2,-1,1)} = i + j(-4+1) + 3k(1(-1))$

$= i-3j-3k$

$D.D = \frac{(i-3j-3k)(i+2j+2k)}{3} = \frac{1-6-6}{3} = -11/3$

Q9: Find D.D of $\phi = x^2-y^2+2z^2$ at pt $(1,2,3)$ in direction of line PQ where $P(1,2,3)$ $Q(5,0,4)$.

Sol: $\vec{a} = \vec{PQ} = \vec{OQ} - \vec{OP} = (5i+0j+4k) - (i+2j+3k)$

$\vec{a} = 4i-2j+k$, $|\vec{a}| = \sqrt{16+4+1} = \sqrt{21}$

$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2-y^2+2z^2)$

$= i(2x) + j(-2y) + k(4z)$

$\nabla\phi_{(1,2,3)} = 2i-4j+12k$

$D.D = \frac{(2i-4j+12k)(4i-2j+k)}{\sqrt{16+4+1}} = \frac{8+8+12}{\sqrt{21}}$

$D.D = \frac{28}{\sqrt{21}}$

Divergence, Curl Of Vector Field

Divergence: If $V(x,y,z)$ is a Continuously differentiable Vector form, is a divergence curve of vector field.

$$\nabla \cdot V = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right] \cdot \left[V_1 i + V_2 j + V_3 k \right]$$

$$= \frac{\partial}{\partial x} V_1 + \frac{\partial}{\partial y} V_2 + \frac{\partial}{\partial z} V_3 .$$

The Scalar Quantity $\nabla \cdot V$ is Known as divergence of vector field 'V' and is denoted as "div V".

Note: A vector point function 'V' is said to be Solenoidal if div V=0.

Some Properties Of Divergence:

The divergence of constant vector is zero.

The, div $(V_1 + V_2) = \text{div } V_1 + \text{div } V_2$, where

$$V_1 = V_1 (x, y, z)$$
$$V_2 = V_2 (x, y, z)$$

If $\phi = \phi(x, y, z)$ is a scalar point function and $V = V(x,y,z)$ is a vector point function, then

$$\text{div } (V\phi) = \phi \text{ div } V + V \cdot \text{grad } \phi .$$

If $\vec{r} = xi + yj + zk$, then S.T $\nabla \cdot r^n = n r^{n-2} \cdot \vec{r}$.

Given,

$\vec{r} = xi + yj + zk$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} 2x = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z = \frac{z}{r}$

$\nabla \cdot r^n = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right] r^n$

$= i\frac{\partial r^n}{\partial x} + j\frac{\partial r^n}{\partial y} + k\frac{\partial r^n}{\partial z} = \left[i\, n \cdot r^{n-1} \frac{\partial r}{\partial x} + j \cdot n r^{n-1} \frac{\partial r}{\partial y} + k \cdot n \cdot r^{n-1} \frac{\partial r}{\partial z} \right]$

$\log^n$

n^{n-1}

$= n \cdot r^{n-1} \left[i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z} \right]$

$= n \cdot r^{n-1} \left[\frac{ix}{r} + \frac{jy}{r} + \frac{kz}{r} \right] = \frac{n r^{n-1}}{r} (xi + yj + z$

$= n \cdot r^{n-2} \cdot \vec{r}$

Prove that $\nabla^2 r^n = n(n+1) r^{n-2}$

Let $\vec{r} = xi + yj + zk$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}.$$

$$\frac{\partial r}{\partial y} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2y = \frac{y}{\sqrt{x^2+y^2+z^2}} = \frac{y}{r}.$$

$$\frac{\partial r}{\partial z} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \cdot 2z = \frac{z}{\sqrt{x^2+y^2+z^2}} = \frac{z}{r}.$$

$$\nabla^2 r^n = \nabla \cdot \nabla r^n = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) r^n$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) r^n$$

Consider $\frac{\partial^2}{\partial x^2}(r^n) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} r^n\right)$

$$= \frac{\partial}{\partial x}\left(n \cdot r^{n-1} \frac{\partial r}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(n \cdot r^{n-1} \cdot \frac{x}{r}\right) = \frac{\partial}{\partial x}\left(n \cdot r^{n-2} \cdot x\right)$$

$$= n(n-2) r^{n-3} \frac{\partial r}{\partial x} \cdot x + n \cdot r^{n-2} \cdot 1$$

$$= n(n-2) r^{n-3} \frac{x}{r} \cdot x + n \cdot r^{n-2}$$

$$\therefore \frac{\partial^2 r^n}{\partial x^2} = n(n-2) r^{n-4} x^2 + n r^{n-2} \longrightarrow (1)$$

Illy $$\frac{\partial^2 r^n}{\partial y^2} = n(n-2) r^{n-4} y^2 + n r^{n-2} \longrightarrow (2)$$

$$\frac{\partial^2 r^n}{\partial z^2} = n(n-2) r^{n-4} z^2 + n r^{n-2} \longrightarrow (3)$$

$$\nabla^2 r^n = n(n-2)r^{n-4}x^2 + nr^{n-2} + n(n-2)r^{n-4}y^2 + nr^{n-2} + n(n-2)r^{n-4}z^2 + nr^{n-2}$$

$$= n(n-2)r^{n-4}(x^2+y^2+z^2) + 3r^{n-2} \cdot n$$

$$= n(n-2)r^{n-4} \cdot r^2 + 3nr^{n-2}$$

$$= n(n-2)r^{n-2} + 3nr^{n-2} = nr^{n-2}(n-2+3)$$

$$\nabla^2 r^n = n(n+1)r^{n-2}$$

Q3: Prove that $\mathrm{div}\,\frac{\vec{r}}{r} = \frac{2}{r}$.

Sol: Let $\vec{r} = xi + yj + zk$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\mathrm{div}\,\frac{\vec{r}}{r} = \nabla \cdot \frac{\vec{r}}{r}$$

$$= \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[\frac{xi}{r} + \frac{yj}{r} + \frac{zk}{r}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \left(\frac{\partial}{\partial y}\cdot\frac{y}{r}\right) + \left(\frac{\partial}{\partial z}\cdot\frac{z}{r}\right)$$

Consider $\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{r - \frac{\partial r}{\partial x}\cdot x}{r^2} = \frac{r}{r^2} - \frac{\partial r}{\partial x}\frac{x}{r^2}$

$$= \frac{1}{r} - \frac{x^2}{r^3} \longrightarrow (1)$$

Similarly,

$$\frac{\partial}{\partial y}\left(\frac{y}{r}\right) = \frac{1}{r} - \frac{y^2}{r^3} \longrightarrow (2)$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r}\right) = \frac{1}{r} - \frac{z^2}{r^3} \longrightarrow (3)$$

$$\frac{\partial}{\partial x}\cdot\frac{x}{r}+\frac{\partial}{\partial y}\cdot\frac{y}{r}+\frac{\partial}{\partial y}\frac{z}{r} = \frac{3}{r} - \frac{(x^2+y^2+z^2)}{r^3}$$

$$= \frac{3}{r}-\frac{r^2}{r^3} = \frac{3}{r}-\frac{1}{r} = \frac{2}{r}.$$

$$\text{div}\,\frac{\vec{r}}{r} = \frac{2}{r}.$$

Q4: Prove that $\text{div}(r^n \vec{r}) = (n+3)r^n$

Sol: $\vec{r} = xi+yj+zk$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\text{div}\, r^n\vec{r} = \left[i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right] r^n(xi+yj+zk)$

$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(r^n xi+r^n yj+r^n zk)$

$= \frac{\partial}{\partial x}(r^n x)+\frac{\partial}{\partial y}(r^n y)+\frac{\partial}{\partial z}(r^n z)$

Consider $\frac{\partial}{\partial x}(r^n x) = nr^{n-1}\frac{\partial r}{\partial x}x+(1)(r^n)$

$= nr^{n-1}\frac{x}{r}\cdot x+r^n$

$\frac{\partial}{\partial x}(r^n x) = nr^{n-2}x^2+r^n \longrightarrow (1)$

$\frac{\partial}{\partial y}(r^n y) = nr^{n-2}y^2+r^n \longrightarrow (2)$

$\frac{\partial}{\partial z}(r^n z) = nr^{n-2}z^2+r^n \longrightarrow (3)$

$\frac{\partial}{\partial x}(xr^n)+\frac{\partial}{\partial y}(r^n y)+\frac{\partial}{\partial z}(r^n z) = nr^{n-2}(x^2+y^2+z^2)+3r^n$

$= nr^{n-2}r^2+3r^n$

$= nr^n+3r^n = r^n(n+3)$

$\text{div}\, r^n\vec{r} = (n+3)r^n$

Q5: Prove that $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$.

Sol: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla^2 f(r) = \nabla \cdot \nabla f(r)$

$= \nabla \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) f(r)$

$= \frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r)$

Consider $\frac{\partial^2}{\partial x^2} f(r) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x} f(r)\right]$

$= \frac{\partial}{\partial x}\left(f'(r)\frac{\partial r}{\partial x}\right] = \frac{\partial}{\partial x}\left[f'(r)\cdot\frac{x}{r}\right]$

$= f''(r)\frac{\partial r}{\partial x}\cdot\frac{x}{r} + f'(r)\left(\frac{1}{r}\right) + f'(r)\cdot x\left(\frac{-1}{r^2}\right)$

$\frac{\partial^2 f(r)}{\partial x^2} = f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{1}{r}\right) - f'(r)\frac{x^2}{r^3} \longrightarrow (1)$

$\text{lly.} \quad \frac{\partial^2}{\partial y^2} f(r) = f''(r)\frac{y^2}{r^2} + \frac{f'(r)}{r} - f'(r)\frac{y^2}{r^3} \longrightarrow (2)$

$\frac{\partial^2}{\partial z^2} f(r) = f''(r)\frac{z^2}{r^2} + \frac{f'(r)}{r} - f'(r)\frac{z^2}{r^3} \longrightarrow (3)$

Add (1), (2) & (3),

$\frac{\partial^2}{\partial x^2} f(r) + \frac{\partial^2}{\partial y^2} f(r) + \frac{\partial^2}{\partial z^2} f(r) = \left[\frac{f''(r)}{r^2}(x^2+y^2+z^2) - \frac{f'(r)}{r^3}(x^2+y^2+z^2)\right] + \frac{3}{r} f'(r)$

$= f''(r)\frac{r^2}{r^2} + \frac{3}{r} f'(r) - \frac{f'(r)}{r^3}\cdot r^2$

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r).$$

Q6: Prove that $\operatorname{div} \frac{\vec{r}}{r^3} = 0$ and $\operatorname{div} \vec{r} = 3$.

Sol: $\vec{r} = xi + yj + zk$

$r = |r| = \sqrt{x^2+y^2+z^2}$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$.

$$\operatorname{div} \frac{\vec{r}}{r^3} = \frac{\nabla \cdot r}{r^3} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\frac{\vec{r}}{r^3}$$

$$= \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[\frac{xi+yj+zk}{r^3}\right]$$

$$= \frac{\partial}{\partial x} \cdot \frac{x}{r^3} + \frac{\partial}{\partial y} \cdot \frac{y}{r^3} + \frac{\partial}{\partial z} \cdot \frac{z}{r^3}$$

Consider $\frac{\partial}{\partial x} \cdot \frac{x}{r^3} = \frac{r^3(1) - x \cdot 3r^2 \cdot \frac{\partial r}{\partial x}}{(r^3)^2} = \frac{r^3 - x \cdot 3r^2 \cdot \frac{x}{r}}{r^6}$

$$= \frac{r^3 - 3x^2 r}{r^6} = \frac{1}{r^3} - \frac{3rx^2}{r^6}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) = \frac{1}{r^3} - \frac{3x^2}{r^5} \longrightarrow (1)$$

$$\text{llly } \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) = \frac{1}{r^3} - \frac{3y^2}{r^5} \longrightarrow (2)$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{1}{r^3} - \frac{3z^2}{r^5} \longrightarrow (3)$$

Add (1), (2) & (3),

$$\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right) = \frac{3}{r^3} - \frac{3}{r^5}(x^2+y^2+z^2)$$

$$= \frac{3}{r^3} - \frac{3}{r^5} \cdot r^2 = \frac{3}{r^3} - \frac{3}{r^3} = 0.$$

$$\text{div} \frac{\vec{r}}{r^3} = 0.$$

Q7: If $\bar{a}$ (Vector) is a Constant Vector, then Prove that

$$\nabla(\vec{a} \cdot \vec{r}) = \vec{a}.$$

Sol: Let $\vec{a} = i a_1 + a_2 j + a_3 k$, $\vec{r} = xi + yj + zk$

$$\vec{a} \cdot \vec{r} = (a_1 i + a_2 j + a_3 k) \cdot (xi + yj + zk)$$

$$= a_1 x + a_2 y + a_3 z$$

$$\nabla(\vec{a} \cdot \vec{r}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot (a_1 x + a_2 y + a_3 z)$$

$$= a_1 i + a_2 j + a_3 k = \vec{a}.$$

$$\nabla(\vec{a} \cdot \vec{r}) = \vec{a}.$$

Q8: If $V = (x+3y)i + (y-3z)j + (x-2z)k$, then S.T 'V' is Solenoidal.

Sol: $$\nabla \cdot V = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[(x+3y)i + (y-3z)j + (x-2z)k\right]$$

$$= \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(x-2z)$$

$$= 1 + 1 - 2 = 0.$$

$$\therefore \nabla \cdot V = 0$$

$\therefore$ 'V' is Solenoidal.

Q9: Find m if $\vec{a} = (x+2y)i + (my+4z)j + (5z+6x)k$ is Solenoidal.

Sol:- Since $\vec{a}$ is a solenoidal.

$$\nabla \cdot \vec{a} = 0.$$

$$\left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[(x+2y)i + (my+4z)j + (5z+6x)k\right] = 0$$

$$\Rightarrow \frac{\partial}{\partial x}(x+2y) + \frac{\partial}{\partial y}(my+4z) + \frac{\partial}{\partial z}(5z+6x) = 0$$

$$1 + m + 5 = 0$$

$$m + 6 = 0$$

$$\boxed{m = -6.}$$

CURL: If $V(x,y,z)$ is continuously differential vector point function then "$\nabla \times V$" is called "curl of V."

$$\left(\nabla \times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}\right)$$

where $V = V_1 i + V_2 j + V_3 k$.

Note: A vector point function 'V' is said to be irrotational if curl $\vec{V} = 0$.

SOME IMPORTANT PROPERTIES OF CURL:

→ If 'V' is a constant vector, then curl $V = 0$.

→ $\text{Curl}(\alpha_1 V_1 + \alpha_2 V_2) = \alpha_1 \text{curl} V_1 + \alpha_2 \text{curl} V_2$, where $V_1 = V_1(x,y,z)$, $V_2 = V_2(x,y,z)$

→ $\text{Curl}(\phi V) = (\text{grad}\,\phi) \times V + \phi\, \text{curl} V$

→ $\text{Curl}(V_1 \times V_2) = V_1(\nabla V_2) - V_2(\nabla \cdot V_1) + (V_2 \cdot \nabla)V_1 - (V_1 \cdot \nabla)V_2$.

Q1: If 'V' is a vector point function, then S.T $div(curl V) = 0$.

Sol: Let $\underline{V} = V_1 i + V_2 j + V_3 k$

$$curl\,V = \nabla \times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i\left(\partial/\partial y\, V_3 - \partial/\partial z\, V_2\right) - j\left(\partial/\partial x\, V_3 - \partial/\partial z\, V_1\right) + k\left(\partial/\partial x\, V_2 - \partial/\partial y\, V_1\right)$$

$$= i\left(\partial/\partial y\, V_3 - \partial/\partial z\, V_2\right) + j\left(\partial/\partial z\, V_1 - \frac{\partial V_3}{\partial x}\right) + k\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)$$

$$div(curl V) = \nabla \cdot (\nabla \times V)$$

$$= \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[i\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) + j\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) + k\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right]$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \cdot \partial x} + \frac{\partial^2 V_2}{\partial z \cdot \partial x} - \frac{\partial^2 V_1}{\partial x \partial y}$$

$$div(curl V) = 0.$$

Q2: If 'V' is a vector point function, then S.T $curl(curl V) = \nabla(\nabla \cdot V) - \nabla^2 V$.

Sol: Let $V = V_1 i + V_2 j + V_3 k$

$$curl V = \nabla \times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= i\left(\frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z}\right)+j\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)+k\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)$$

$$\text{curl}(\text{curl}\,v) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial v_3}{\partial y}-\frac{\partial v_2}{\partial z} & \frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x} & \frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \Sigma i\left\{\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x}-\frac{\partial v_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z}-\frac{\partial v_3}{\partial x}\right)\right\}$$

$$= \Sigma i\left[\frac{\partial^2 v_2}{\partial y\cdot\partial x}-\frac{\partial^2 v_1}{\partial y^2}-\frac{\partial^2 v_1}{\partial z^2}+\frac{\partial^2 v_3}{\partial z\cdot\partial x}\right]$$

$$= \Sigma i\left[\frac{\partial^2 v_2}{\partial y\cdot\partial x}-\frac{\partial^2 v_1}{\partial y^2}-\frac{\partial^2 v_1}{\partial z^2}+\frac{\partial^2 v_1}{\partial x^2}-\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_3}{\partial z\cdot\partial x}\right]$$

$$= \Sigma i\left[-\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)+\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_2}{\partial y\cdot\partial x}+\frac{\partial^2 v_3}{\partial z\cdot\partial x}\right)\right]$$

$$= \Sigma i\left[\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial v_3}{\partial z}\right)-\left(\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+\frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

$$= \Sigma i\left\{\frac{\partial}{\partial x}(\nabla\cdot v)-\nabla^2 v_1\right\}$$

$$= i\frac{\partial}{\partial x}(\nabla\cdot v)-i\nabla^2 v_1+j\frac{\partial}{\partial y}(\nabla\cdot v)-j\nabla^2 v_2+k\frac{\partial}{\partial z}(\nabla\cdot v)-k\nabla^2 v_3$$

$$= \left[i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right]\nabla\cdot v-\left[i\nabla^2 v_1+j\nabla^2 v_2+k\nabla^2 v_3\right]$$

$$\text{curl}(\text{curl}\,v) = \nabla(\nabla\cdot v)-\nabla^2(v).$$

Q6: If $\vec{F} = (x+y+1)i + j - (x+y)k$, S.T $\vec{F} \cdot \text{Curl}\vec{F} = 0$.

Sol: $\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+y+1) & 1 & -(x+y) \end{vmatrix}$

$= i\left[\frac{-\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(1)\right] - j\left[\frac{-\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(x+y+1)\right] + k\left[\frac{\partial}{\partial x}(1) - \frac{\partial}{\partial y}(x+y+1)\right].$

$= i(-1-0) - j(-1-0) + k(0-1) = -i + j - k.$

$\vec{F} \cdot \text{curl}\vec{F} = \left((x+y+1)i + j - (x+y)k\right)(-i + j - k)$

$= -(x+y+1) + 1 + (x+y)$

$= -x - y - 1 + 1 + x + y = 0$

$\therefore \vec{F} \cdot \text{curl}\vec{F} = 0.$

Q7: If $\vec{F} = x^2yi - 2xzj + 2yzk$, then find $\text{div}\vec{F}$, $\text{Curl}\vec{F}$ and $\text{curl}[\text{curl}\,\vec{F}]$.

Sol: Given, $\vec{F} = x^2yi - 2xzj + 2yzk$

$\text{div}\vec{F} = \nabla \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \vec{F}$

$= \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[x^2yi - 2xzj + 2yzk\right]$

$= \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(2xz) + \frac{\partial}{\partial z}(2yz)$

$= 2xy - 0 + 2y$

$= 2y[x+1]$

$$\operatorname{div}\vec{F} = 2xy+0+2y = 2(xy+y)$$

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right) \ldots \frac{\partial}{\partial z}(x^2y) - \frac{\partial}{\partial x}(2yz)\Big)$$

$$+k\Big(\ldots -2x \ldots (x^2y)\Big)$$

$= i\Big(\ldots (+2 \ldots$

curl $\vec{F}$ ... i ...

k

$\partial/\partial z$

$-(2z+x \ldots$

$(2z+x \ldots \; \ldots 2x)$

Q3.

Sol:

Q4:

Sol

If 'V' is a vector point function, then curl[grad φ] = 0.

$$\text{grad}\,\phi = \nabla\phi$$

$$= \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\phi$$

$$\nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\text{curl}[\text{grad}\,\phi] = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial\phi/\partial x & \partial\phi/\partial y & \partial\phi/\partial z \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial\phi}{\partial y}\right] - j\left[\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial z} - \frac{\partial}{\partial z}\cdot\frac{\partial\phi}{\partial x}\right] + k\left[\frac{\partial}{\partial x}\cdot\frac{\partial\phi}{\partial y} - \frac{\partial}{\partial y}\cdot\frac{\partial\phi}{\partial x}\right]$$

$$= \frac{i\,\partial^2\phi}{\partial y\cdot\partial z} - \frac{i\,\partial^2\phi}{\partial z\cdot\partial y} + \frac{j\,\partial^2\phi}{\partial z\cdot\partial x} - \frac{j\,\partial^2\phi}{\partial x\cdot\partial z} + \frac{k\,\partial^2\phi}{\partial x\cdot\partial y} - \frac{k\,\partial^2\phi}{\partial y\cdot\partial x}$$

$$\text{curl}[\text{grad}\,\phi] = 0.$$

Find a, b, c if vector $F = (x+2y+az)i + (bx-3y-z)j + [4x+2z+cy]k$ is irrotational.

: Given, $\vec{F} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$

Since $\vec{F}$ is irrotational, then curl $\vec{F} = 0$

$$\nabla\times\vec{F} = 0.$$

i.e

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$$= i\left[\frac{\partial}{\partial y}(4x+cy+az)-\frac{\partial}{\partial z}(bx-3y-z)\right]-j\left[\left(\frac{\partial}{\partial x}(4x+cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right)\right]+k\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]$$

$$\Rightarrow i(c+1)-j(4-a)+k(b-2)=0.$$

$$i(c+1)+j(a-4)+k(b-2)=0$$

$$\therefore a=4,\ b=2,\ c=-1.$$

Q5: S.T $\vec{F}=(\sin y+z)i+(x\cos y-z)j+(x-y)k$ is irrotational.

Sol: $\text{Curl}\,\vec{F}=\nabla\times\vec{F}$

$$=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\sin y+z) & (x\cos y-z) & (x-y) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]-j\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(\sin y+z)\right]+k\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(\sin y+z)\right]$$

$$= i(-1-(-1))-j(1-1)+k(\cos y-\cos y)$$

$$=0.$$

$\therefore \vec{F}$ is irrotational.

Potential Function

→ Given $\vec{F}$ is a vector point function, if there exists a scalar point function ϕ such that $F = \nabla\phi$, then ϕ is said to be scalar point function of "F".

In such case, curl $\vec{F} = 0$ i.e $\nabla \times \vec{F} = 0$.

i.e $\vec{F}$ is an irrotational.

→ If $\vec{F}$ is irrotational, then there exists a scalar potential function ϕ such that $F = \nabla\phi$

$$\text{i.e } F_1 i + F_2 j + F_3 k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

Equating like vectors,

$$F_1 = \frac{\partial\phi}{\partial x}, \quad F_2 = \frac{\partial\phi}{\partial y}, \quad F_3 = \frac{\partial\phi}{\partial z}.$$

The scalar potential function ϕ is determined by the above relation by using exact differentials i.e,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \text{ and}$$

Integrate.

Q1: Show that the following Vector $(x^2-yz)i+(y^2-zx)j+(z^2-xy)k$ is irrotational. Find the Scalar potential function.

Sol: Given, $\vec{F} = (x^2-yz)i + (y^2-zx)j + (z^2-xy)k$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2-yz) & (y^2-zx) & (z^2-xy) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(z^2-xy) - \frac{\partial}{\partial z}(y^2-zx)\right] + j\left[\frac{\partial}{\partial z}(x^2-yz) - \frac{\partial}{\partial x}(z^2-xy)\right] + k\left[\frac{\partial}{\partial x}(y^2-zx) - \frac{\partial}{\partial y}(x^2-yz)\right]$$

$$= i\left[-x-(-x)\right] + j\left[-y-(-y)\right] + k\left[-z-(-z)\right]$$

$= 0.$

$\therefore \vec{F}$ is irrotational.

If $\vec{F}$ is an irrotational, then $F = \nabla\phi$

$$(x^2-yz)i + (y^2-zx)j + (z^2-xy)k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

Equating like vectors,

$$x^2-yz = \frac{\partial\phi}{\partial x}, \quad y^2-zx = \frac{\partial\phi}{\partial y}, \quad z^2-xy = \frac{\partial\phi}{\partial z}$$

W.K.T $d\phi = \frac{\partial\phi}{\partial x}.dx + \frac{\partial\phi}{\partial y}.dy + \frac{\partial\phi}{\partial z}.dz$

$$d\phi = (x^2-yz)dx + (y^2-zx)dy + (z^2-xy)dz$$

Integrating on both sides,

$d\phi = x^2dx + y^2dy + z^2dz - (yzdx \ldots$

$d\phi = x^2dx + y^2dy + z^2dz \ldots$

$$\int d\phi = \int (x^2 - yz)\,dx + \int (y^2 - zx)\,dy + \int (z^2 - xy)\,dz$$

$$\phi = \frac{x^3}{3} - yzx + \frac{y^3}{3} - zxy + \frac{z^3}{3} - xyz$$

$$\phi = \frac{x^3 + y^3 + z^3}{\cancel{3}} - \cancel{3}xyz.$$

Find the Constants a, b, c of Vector $V = (x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$. Find ϕ such that $V = \nabla\phi$

Given $\vec{V}$ is irrotational,

$$\text{curl } \vec{V} = 0$$

$$\nabla \times \vec{V} = 0$$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x + 2y + az) & (bx - 3y - z) & (4x + cy + 2z) \end{vmatrix} = 0$$

$$= i\left[\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z)\right] + j\left[\frac{\partial}{\partial z}(x + 2y + az) - \frac{\partial}{\partial x}(4x + cy + 2z)\right] + k\left(\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az)\right)$$

$$= i(c + 1) + j(4 - a) + k(b - 2) = 0$$

$c + 1 = 0$ $\qquad$ $4 - a = 0$ $\qquad$ $b - 2 = 0$

$c = -1$ $\qquad$ $a = 4$ $\qquad$ $b = 2$

$$V = (x + 2y + 4z)i + (2x - 3y - z)j + (4x - y + 2z)k$$

$V = \nabla\phi$

$(x+2y+4z)i + (2x-3y-z)j + k(4x-y+2z) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$

Equating like vectors,

$x+2y+4z = \frac{\partial\phi}{\partial x}$, $2x-3y-z = \frac{\partial\phi}{\partial y}$, $4x-y+2z = \frac{\partial\phi}{\partial z}$.

W.K.T $d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}\cdot dy + \frac{\partial\phi}{\partial z}\cdot dz$

$d\phi = (x+2y+4z)dx + (2x-3y-z)dy + (4x-y+2z)dz$

Integrating on both sides,

$\int d\phi = \int(x+2y+4z)dx + \int(2x-3y-z)dy + \int(4x-y+2z)dz$

$\phi = \frac{x^2}{2} + 2xy + 4xz + 2xy - \frac{3y^2}{2} - zy + 4xz - yz + \frac{2z^2}{2}$.

$= \frac{x^2}{2} - \frac{3y^2}{2} + \frac{2z^2}{2} + 4xy + 8xz - 2yz$

$\phi = \frac{x^2 - 3y^2 + 2z^2}{2} + 2(2xy + 4xz\} - yz)$.

Q3. S.T $\vec{F} = yzi + zxj + xyk$ is irrotational and find ϕ so that $\vec{F} = \nabla\phi$.

Sol: Given, $\vec{F} = yzi + zxj + xyk$

$$\text{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= i\left[\partial/\partial y(xy) - \partial/\partial z(zx)\right] + j\left[\frac{\partial}{\partial z}(yz) - \frac{\partial}{\partial x}(xy)\right] + k\left[\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right]$$

$$= i(x-x) + j(-y+y) + k(z-z) = 0.$$

$\therefore \vec{F}$ is irrotational.

$$\vec{F} = \nabla\phi$$

$$(yz)i + (zx)j + (xy)k = \left[i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}\right]$$

$$\frac{\partial\phi}{\partial x} = yz, \quad \frac{\partial\phi}{\partial y} = zx, \quad \frac{\partial\phi}{\partial z} = xy.$$

W.K.T $\quad d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}\cdot dy + \frac{\partial\phi}{\partial z}\cdot dz$

$$d\phi = yz\,dx + zx\,dy + xy\,dz. = \int d(xyz)$$

Integrating on both sides,

$$\int d\phi = \int yz\,dx + \int zx\,dy + \int xy\,dz =$$

$$\phi = xyz + xyz + xyz$$

$$\phi = xyz.$$

Q4: S.T $\vec{F} = (y^2\cos x + z^3)i + (2y\sin x - 4)j + (3xz^2+2)k$ is Conservati[ve] and find its scalar potential.

Sol: Given,

$$\vec{F} = (y^2\cos x + z^3)i + (2y\sin x - 4)j + (3xz^2+2)k$$

$$\text{curl}\,\vec{F} = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y^2\cos x + z^3) & (2y\sin x - 4) & (3xz^2+2) \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}(3xz^2+2)-\frac{\partial}{\partial z}(2y\sin x-4)\right]+j\left[\frac{\partial}{\partial z}(y^2\cos x+z^3)-\frac{\partial}{\partial x}(3xz^2+2)\right]$$

$$+k\left[\frac{\partial}{\partial x}(2y\sin x-4)-\frac{\partial}{\partial y}(y^2\cos x+z^3)\right]$$

$$= 0i-j(3z^2-3z^2)+k(2y\cos x-2y\cos x)=0$$

$\therefore \vec{F}$ is conservative.

$$\vec{F}=\nabla\phi$$

$$(y^2\cos x+z^3)i+(2y\sin x-4)j+(3xz^2+2)k=\left[i\frac{\partial\phi}{\partial x}+j\frac{\partial\phi}{\partial y}+k\frac{\partial\phi}{\partial z}\right]$$

$$\frac{\partial\phi}{\partial x}=y^2\cos x+z^3,\ \frac{\partial\phi}{\partial y}=2y\sin x-4,\ \frac{\partial\phi}{\partial z}=3xz^2+2.$$

W.K.T, $d\phi=\frac{\partial\phi}{\partial x}\cdot dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}\cdot dz$

$$d\phi=(y^2\cos x+z^3)dx+(2y\sin x-4)dy+(3xz^2+2)dz$$

Integrating on both sides,

$$\int d\phi=\int(y^2\cos x+z^3)dx+\int 2y\sin x\,dy-4\int dy+3x\int z^2dz+2\int dz$$

$$=y^2\sin x+z^3x+2\sin x\frac{y^2}{2}-4y+3x\cdot\frac{z^3}{3}+2z$$

$$=2\sin x y^2+2z^3x-4y+2z.$$

$$\phi=2(y^2\sin x+z^3x-2y+z).$$

(or)

$$\int d\phi=\int y^2\cos x\,dx+\int z^3dx+\int 2y\sin x\,dy-4\int dy+3x\int z^2dz+2\int dz$$

$$=\int(y^2\cos x\,dx+2y\sin x\,dy)+\int(z^3dx+3xz^2dz)-4\int dy+2\int dz$$

$$=\int d(y^2\sin x)+\int d(z^3x)-4\int dy+2\int dz$$

$$\phi=y^2\sin x+z^3x-4y+2z.$$

LINE, SURFACE AND VOLUME INTEGRAL

LINE INTEGRAL: Any integral which is to be evaluated along a curve is called a line Integral. Let $F(x,y,z) = F_1 i + F_2 j + F_3 k$ be a continuous vector point function at every point of curve in space. It is denoted by $\oint_C \vec{F} \cdot \vec{dr}$

Note: $\vec{F} = F_1 i + F_2 j + F_3 k$, $\vec{r} = xi + yj + zk$.

① → $\vec{dr} = dx i + dy j + dz k$.

$$\int_C \vec{F} \cdot \vec{dr} = \int_C F_1 dx + F_2 dy + F_3 dz.$$

② → If a curve is in parametric form, $x = x(t), y = y(t)$, $z = z(t)$, let the end points of curve $t = t_1$ at A, $t = t_2$ at B.

then
$$\int_C \vec{F} \cdot \vec{dr} = \int_{t_1}^{t_2} \left[F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right] dt.$$

③ → If "c" is closed curve, then Integral is defined by

$$\oint_C \vec{F} \cdot \vec{dr}.$$

④ → If "c" is simple closed curve, then the Tangent line ($\int \vec{F}$) integral of $\vec{F}$ vector around 'c' is called circulation of c vector.

$$\oint_C \vec{F} \cdot \vec{dr} = \oint_C F_1 dx + F_2 dy + F_3 dz.$$

If $\oint_C \vec{F} \cdot \vec{dr} = 0$, then $\vec{F}$ is irrotational.

⑤ → Work done by the Force:

If F represents the force acting on a particle moving along the arc AB, the work done during the displacement of δr is $\quad \delta r = \overrightarrow{F} \cdot \overrightarrow{dr}$.

The total work done by the force during the displacement from A to B

$$\int_A^B \overrightarrow{F} \cdot \overrightarrow{\delta r} = \int_A^B \overrightarrow{F} \cdot \overrightarrow{dr}.$$

③ → The Line Integral is independent of path if $\overrightarrow{F}$ is Conservative field or irrotational, then $F = \nabla\phi$.

Q1: If $\overrightarrow{F} = [5xy - 6x^2]i + [2y - 4x]j$. Evaluate $\int_C \overrightarrow{F} \cdot \overrightarrow{dr}$ along the Curve 'c' in xy-plane, $y = x^3$ from point (1,1) to (2,8).

Sol: Given, $\overrightarrow{r} = xi + yj + zk$

$$\overrightarrow{dr} = dxi + dyj + dzk$$

$$\int_C \overrightarrow{F} \cdot \overrightarrow{dr} = \int_C \{(5xy - 6x^2)i + (2y - 4x)j\} \{dxi + dyj + dzk\}$$

$y = x^3$

$dy = 3x^2 dx$

$$= \int_C \{(5x \cdot x^3 - 6x^2)i + (2x^3 - 4x)j\} \{dxi + dyj + dzk\}$$

$$= \int_C (5x^4 - 6x^2)\,dx + (2x^3 - 4x)\,dy$$

$$= \int_C \{5x^4 - 6x^2 + (2x^3 - 4x)3x^2\}\,dx$$

$$= \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

$$= \left[\left(\frac{5x^5}{5}\right) - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4}\right]_1^2$$

$= (32-1) - [16-2] + [64-1] - 3[16-1]$

$= 31 - 14 + 63 - 45 = 35.$

$\int_C \vec{F}\cdot\vec{dr} = 35.$

Q2: Evaluate $\int_C \vec{F}\cdot\vec{dr}$ where $\vec{F} = x^2 i + y^3 j$ and curve 'C' is arc of Parabola $y = x^2$ in xy-plane from $(0,0)$ to $(1,1)$.

Sol: Given, $\vec{F} = x^2 i + y^3 j$

$\vec{dr} = dx i + dy j + dz k$

$\int_C \vec{F}\cdot\vec{dr} = \int_C (x^2 i + y^3 j)(dx i + dy j + dz k)$

$= \int_C x^2 dx + y^3 dy$

$y = x^2$

$dy = 2x\,dx$

$\int_C \vec{F}\cdot\vec{dr} = \int_C x^2 dx + (x^2)^3 2x\,dx = \int_0^1 (x^2 + 2x^7)\,dx$

$= \left(\frac{x^3}{3}\right)_0^1 + \left(\frac{2x^8}{8}\right)_0^1 = \frac{1}{3} + \frac{2}{8} = \frac{8+6}{24} = \frac{7}{12}.$

Q3: If $\vec{F} = (3x^2+6y)i - 14yz j + 20xz^2 k$, then Evaluate $\int_C \vec{F}\cdot\vec{dr}$, where 'C' is a straight line joining the points $(0,0,0)$ to $(1,1,1)$.

Sol: let $A(0,0,0)$ $B(1,1,1)$

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t.$

$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$

$x = y = z = t$

$dx = dy = dz = dt$

$\int_C \vec{F}\cdot\vec{dr} = \int_C ((3x^2+6y)i - (14yz)j + (20xz^2)k)\cdot(dx\,i + dy\,j + dz\,k)$

$= \int_C (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz$

$= \int_C (3t^2+6t)dt - 14t^2 dt + 20t^3 dt$

$= \int_C (-11t^2+6t+20t^3)\,dt$

$= \int_0^1 (20t^3 - 11t^2 + 6t)dt = \left(\frac{20t^4}{4} - \frac{11t^3}{3} + \frac{6t^2}{2}\right)_0^1$

$= 5(1) - \frac{11}{3}(1) + 3(1) = 13/3.$

Q4: Find the work done by the force field $\vec{F} = (x^2-xy)i + (y^2-2xy)j$ in moving the particle from point $(1,1)$ to $(2,4)$ along the curve $y = x^2$.

Sol: Work done $\delta r = \vec{F}\cdot\vec{dr}$

$= [(x^2-xy)i + (y^2-2xy)j]\cdot[dx\,i + dy\,j + dz\,k]$.

$= (x^2-xy)dx + (y^2-2xy)dy.$

$\rightarrow y = x^2$

$dy = 2x\,dx.$

$= (x^2 - x\cdot x^2)dx + (x^4 - 2x\cdot x^2)\,2x\,dx$

$= (x^2 - x^3 + 2x^5 - 4x^4)\,dx = \int_1 (x^2 - x^3 + 2x^5 - 4x^4)$

On Integrating,

$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{2x^6}{6} - \frac{4x^5}{5}\right]_1^2$

$= \left[\frac{8}{3} - \frac{16}{4} + \frac{2\times 64}{6} - \frac{4\times 32}{5}\right] - \left(\frac{1}{3} - \frac{1}{4} + \frac{2}{6} - \frac{4}{5}\right)$

$= \frac{8}{3} - 4 + \frac{64}{3} - \frac{128}{5} - \frac{1}{3} + \frac{1}{4} - \frac{1}{3} + \frac{4}{5}$

$= \frac{6}{3} + \frac{64}{3} - \frac{128}{5} - \frac{15}{4} + \frac{4}{5} = \frac{180 + 1280 - 1536}{60}$

$= \frac{-313}{60}$.

Q5: Evaluate $\int \vec{F}\cdot\vec{dr}$ where $\vec{F} = (x^2+y^2)i - 2xyj$, C is the rectangle in xy-plane and bounded by $y=0,\ x=0,\ x=a,\ y=b$.

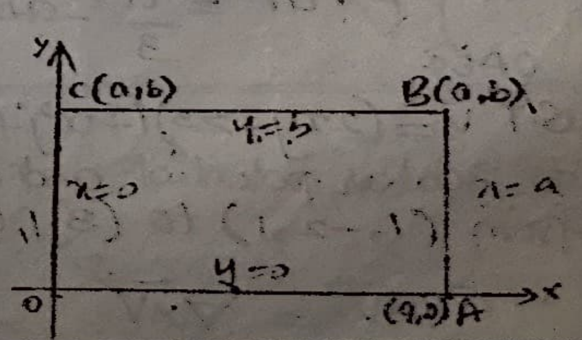
Sol: $\int_C \vec{F}\cdot\vec{dr} = \int_C \{(x^2+y^2)i - (2xy)j\}\{dx\,i + dy\,j\}$

$= \int_C (x^2+y^2)dx - 2xy\,dy \longrightarrow (1)$

Along $\overrightarrow{OA}$: $y=0$

$dy = 0$, $x: 0 \to a$

$\int_{OA} \vec{F}\cdot dr = \int_{OA} x^2 dx$



$$= \int_0^a x^2 dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3} \longrightarrow (2)$$

Along $\overrightarrow{AB}$: $x = a$ $\quad y: 0 \to b$
$dx = 0$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{AB} -2ay\,dy = -2a\int_0^b y\,dy = -2a\left[\frac{y^2}{2}\right]_0^b$$

$$= \frac{-2ab^2}{2} = -ab^2.$$

Along $\overrightarrow{BC}$: $y = b$ $\quad x: a \to 0$
$dy = 0$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_{BC} (x^2+y^2)\,dx = \int_a^0 (x^2+b^2)\,dx = \left(\frac{x^3}{3} + b^2x\right)_a^0$$

$$= \frac{-a^3}{3} - ab^2.$$

Along $\overrightarrow{CD}$: $x = 0$ $\quad y: b \to 0$
$dx = 0$

$$\int_{CD} \vec{F}\cdot\vec{dr} = \int_{CD} -2axy\,dy = 0.$$

$$\therefore \int_{OABC} \vec{F}\cdot\vec{dr} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -2ab^2.$$

Q6: S.T $\vec{V} = (2xy+z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is conservative field. Find its scalar potential and also the work done in moving from $(1,-2,1)$ to $(3,1,4)$.

Sol: $\quad \text{Curl}\,\vec{V} = \nabla \times \vec{V}$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (2xy+z^3) & x^2 & 3z^2x \end{vmatrix}$$

$$= i(0-0) + j(3z^2-3z^2) + k(2x-2x)$$

$$= 0.$$

$\text{curl}\,\vec{V} = 0.$

$\therefore \vec{V}$ is a Conservative field.

If $\vec{V}$ is Conservative field then, $\vec{V} = \nabla\phi$

$$(2xy+z^3)i + x^2 j + 3z^2 x k = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}.$$

Equating like Vectors,

$$\frac{\partial\phi}{\partial x} = 2xy+z^3, \quad \frac{\partial\phi}{\partial y} = x^2, \quad \frac{\partial\phi}{\partial z} = 3xz^2$$

W.K.T, $d\phi = \frac{\partial\phi}{\partial x}.dx + \frac{\partial\phi}{\partial y}.dy + \frac{\partial\phi}{\partial z}.dz$

$$= (2xy+z^3)dx + x^2 dy + 3xz^2 dz.$$

$$d\phi = 2xy\,dx + z^3 dx + x^2 dy + 3xz^2 dz$$

Integrating on both sides,

$$\int d\phi = \int d(x^2 y) + \int d(z^3 x)$$

$$\phi = x^2 y + z^3 x.$$

The work done by force $= \int_A^B d\phi = [\phi]_A^B = (x^2y + xz^3)$

$$= (x^2 y + z^3 x)_{(1,-2,1)}^{(3,1,4)} = (9 + (64\times 3)) - (-2+1)$$

$$= 201 + 1 = 202.$$

Surface Integrals

An Integral which is to be evaluated over a surface is called a surface integral.

Suppose 's' is the surface area

Suppose $f(x,y,z)$ is single volume function over the surface 'S'.

Note: To evaluate any surface integral, it is convenient to evaluate double integral of its projection in xy-plane, yz-plane or xz-plane.

$$\rightarrow \iint_S F.n\,ds = \iint_S \vec{F}.ds = \iint_R F.n \frac{dx\,dy}{|n.k|}$$

where 'R' is the projection in xy-plane.

$$\rightarrow \iint_S F.n\,ds = \iint_S \vec{F}.ds = \iint_R F.n \frac{dx\,dz}{|n.j|}$$

where 'R' is the projection in xz-plane.

$$\rightarrow \iint_S F.n\,ds = \iint_S \vec{F}.ds = \iint_R F.n \frac{dy\,dz}{|n.i|}$$

where 'R' is the projection in yz-plane.

Q1: Evaluate $\iint_S A\cdot n\,ds$, where $A = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and where S is the part of plane $2x+3y+6z=12$, which is the first octant.

Sol: Let $\phi = 2x+3y+6z-12 = 0$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$$

$$= 2\hat{i} + 3\hat{j} + 6\hat{k}$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{\sqrt{4+9+36}} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}$$

$$n\cdot k = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\cdot k = \frac{6}{7}$$

$$A\cdot n = (18z\hat{i} - 12\hat{j} + 3y\hat{k})\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)$$

$$\Rightarrow \frac{36z}{7} - \frac{36}{7} + \frac{18y}{7}$$

$$= \frac{1}{7}\left[\cancel{36}^{\,6}\left(\frac{12-2x-3y}{\cancel{6}}\right) - 36 + 18y\right]$$

$2x+3y+6z=12$

$z = \frac{12-2x-3y}{6}$

$$= \frac{(72 - 12x - \cancel{18y} - 36 + \cancel{18y})}{7}$$

$$= \frac{-12x+36}{7} = \frac{6}{7}(6-2x)$$

$$\iint_S A\cdot n\,ds = \iint_S A\cdot\hat{n}\,\frac{dx\,dy}{|n\cdot k|} = \iint_R \frac{1}{7}(36-12x)\frac{dx\,dy}{(6/7)}$$

$$\iint (6-2x)\,dx\,dy$$

$$\int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\, dx\, dy$$

$2x+3y=12$
$y=0$
$x=6$
$y=\frac{12-2x}{3}$

$$= \int_0^6 (6-2x)(y)_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \frac{(6-2x)(12-2x)}{3} dx = \frac{1}{3}\int_0^6 (72-12x-24x+4x^2)\, dx$$

$$= \frac{1}{3}\int_0^6 (4x^2-36x+72)\, dx = \frac{1}{3}\left[\frac{4x^3}{3}-\frac{36x^2}{2}+72x\right]_0^6$$

$$= \frac{1}{3}\left[\frac{4\times 6^3}{3}-\frac{36\times 36}{2}+72\times 6\right] = 24.$$

$$\iint_S A\cdot n\, ds = 24.$$

Q2: Evaluate $\iint \vec{F}\cdot n\, ds$ where $\vec{F} = zi+xj+3y^2zk$ and 's' is the surface of cylinder $x^2+y^2=16$ in the first octant degree, $z=0$ and $z=6$.

Sol: Given, $\vec{F} = zi+xj+3y^2zk$

$$n = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left[i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right](x^2+y^2-16)$$

$$= 2xi+2yj$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2} = 2\times 4 = 8$$

$$\therefore n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj)}{8} = \frac{xi+yj}{4}.$$

$$n\cdot i = \frac{\nabla\phi}{|\nabla\phi|}\cdot i = \frac{xi+yj}{4}\cdot i = \frac{x}{4}.$$

$$\vec{F}\cdot\underline{n} = \left[zi + xj + 3y^2zk\right]\cdot\left[\frac{xi + yj}{4}\right] = \frac{zx + xy}{4} = \frac{x}{4}(z+y).$$

$$\iint \vec{F}\cdot n\,ds = \iint F\cdot n\,\frac{dy\,dz}{|\underline{n}\cdot i|} = \iint \frac{x}{4}(z+y)\,\frac{dy\,dz}{(x/4)}.$$

$$\int_{y=0}^{4}\int_{z=0}^{6}(z+y)\,dy\,dz$$

$x^2+y^2=16$

$y^2=16, y=4$

$$= \int_{y=0}^{4}\left(\frac{z^2}{2} + yz\right)_0^6 dy = \int_0^4 (18+6y)\,dy = \left(18y + \frac{6y^2}{2}\right)_0^4$$

$$= 72 + 48 = 120.$$

$$\iint \vec{F}\cdot n\,ds = 120.$$

Q3: If $\vec{F} = 2yi - zj + x^2k$ and 'S' is the surface of parabolic cylinder and $y^2 = 8x$ in the first octant bounded by the planes $y=4$, $z=6$. Evaluate $\int_S F\cdot n\,ds$.

Sol: Given, $\vec{F} = 2yi - zj + x^2k$

$$n = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[y^2 - 8x\right] = -8i + 2yj$$

$$|\nabla\phi| = \sqrt{64 + 4y^2} = 2\sqrt{16+y^2}$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-8i + 2yj}{2\sqrt{16+y^2}} = \frac{-4i + yj}{\sqrt{16+y^2}}.$$

$$n\cdot i = \frac{-4i + yj}{\sqrt{16+y^2}}\cdot i = \frac{-4}{\sqrt{16+y^2}}.$$

$$\vec{F}\cdot n = (2yi + x^2k - zj)\left(\frac{-4i + yj}{\sqrt{16+y^2}}\right) = \frac{1}{\sqrt{16+y^2}}(-8y - zy)$$

$$\int_S F\cdot n\,ds = \iint F\cdot n \frac{dy\,dz}{|n\cdot i|} = \iint \frac{\frac{1}{\sqrt{16+y^2}}(-8y-zy)\,dy\,dz}{\left(\frac{-4}{\sqrt{16+y^2}}\right)}$$

$$= \iint \frac{-1}{4}(-8y-zy)\,dy\,dz$$

$$= \frac{1}{4}\int_{y=0}^{y=4}\int_{z=0}^{z=6}(8y+zy)\,dy\,dz = \frac{1}{4}\int_{y=0}^{y=4}\left(8yz + \frac{z^2y}{2}\right)_0^6 dy$$

$$= \frac{1}{4}\int_{y=0}^{y=4}\left(48y + \frac{36y}{2}\right)dy = \frac{1}{4}\int_{y=0}^{4}(48y+18y)\,dy$$

$$= \left(\frac{66y^2}{4\times 2}\right)_0^4 = \frac{66\times 16}{4\times 2} = 132.$$

$$\therefore \int F\cdot n\,ds = 132.$$

Q4: Evaluate $\iint_S F\cdot\hat{n}\,ds$ where $\vec{F} = yi + 2xj - zk$ and 's' is the surface of plane $2x+y=6$ in the first octant cut off by the plane $z=4$.

Sol: Given, $\vec{F} = yi + 2xj - zk$

$$2x + y - 6 = 0 \Rightarrow y = 6 - 2x.$$

$$\nabla\phi = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right][2x+y-6] = 2i + j.$$

$$|\nabla\phi| = \sqrt{4+1} = \sqrt{5}.$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2i+j}{\sqrt{5}},\quad n\cdot J = \frac{2i+J}{\sqrt{5}}\cdot J = 1/\sqrt{5}$$

$$\vec{F}\cdot n = (yi+2xj-zk)\left(\frac{2i+j}{\sqrt{5}}\right) = \frac{2y+2x}{\sqrt{5}}.$$

$$\iint_S \vec{F}\cdot n\,ds = \iint_S F\cdot n\frac{dx\,dz}{|n\cdot j|} = \iint \frac{2(x+y)}{\sqrt{5}}\cdot\frac{dx\,dz}{\left(\frac{1}{\sqrt{5}}\right)}$$

$$= 2\iint (x+y)\,dx\,dz = 2\int_{x=0}^{x=3}\int_{z=0}^{z=4}(x+6-2x)\,dx\,dz$$

$$= 2\int_0^3\int_0^4 (6-x)\,dx\,dz = 2\int_0^3 (6z-xz)_0^4\,dx = 2\int_0^3 (24-4x)\,dx$$

$$= 2\left[24x-\frac{\cancel{4}^{2}x^2}{\cancel{2}}\right]_0^3 = 2\left[72-2(9)\right] = 2(54) = 108.$$

$$\iint_S F\cdot\hat{n}\,ds = 108.$$

Q5: Evaluate $\iint_S F\cdot n\,ds$ where $\vec{F} = yzi+zxj+xyk$ and 'S' is a part of surface of sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Sol: $\vec{F} = yzi+zxj+xyk$, $\phi = x^2+y^2+z^2=1$

$$\iint_S F\cdot n\,ds = \iint_R F\cdot n\frac{dx\,dy}{|n\cdot k|}$$

$$n = \frac{\nabla\phi}{|\nabla\phi|}.$$

$$\nabla\phi = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-1)$$

$$= 2xi+2yj+2zk.$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2} = 2\sqrt{1} = 2.$$

$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj+zk)}{2} = xi+yj+zk.$

$n \cdot k = (xi+yj+zk)\cdot k = z$, $\vec{F}\cdot n = [yzi + xzj + xyk][xi+yj+zk]$

$= xyz + xyz + xyz$

$F \cdot n = 3xyz$

$\iint_S F\cdot n\, ds = \iint 3xyz \frac{dx\,dy}{z} = 3\iint xy\, dx\,dy$

$x^2+y^2+z^2=1$

$= 3\iint xy\,dx\,dy = 3\int_0^1\int_0^{\sqrt{1-x^2}} xy\,dx\,dy$

$x^2+y^2=1$

$x=0,\ y=0$

$y=1 \quad x=1$

$= 3\int_0^1 \left(x\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} dx$

$x^2+y^2=1$

$y^2=1-x^2$

$y=0 \quad y=\sqrt{1-x^2}$

$x=0,\ x=1$

$= 3\int_0^1 \frac{x(1-x^2)}{2}dx = \frac{3}{2}\int_0^1 (x-x^3)\,dx$

$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left[\frac{1}{2}-\frac{1}{4}\right] = \frac{3}{2}\frac{(2-1)}{4\times 2} = \frac{3}{8}.$

$\iint_S F\cdot n\, ds = \frac{3}{8}.$

Q6: Evaluate $\iint_S F\cdot n\,ds$ where $\vec{F} = z^2 i + x^2 j - y^2 z k$, and 'S' be the surface of plane $x^2+y^2=16$ in the first Octant cut off by the plane $z=0,\ z=5.$

Sol: Given, $\vec{F} = z^2 i + x^2 j - y^2 z k$

$\phi = x^2+y^2-16$

$n = \frac{\nabla\phi}{|\nabla\phi|}$, $\nabla\phi = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[x^2+y^2-16\right]$

$n = 2xi + 2yj$

$|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2} = 2\sqrt{16} = 2(4) = 8.$

$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(xi+yj)}{8} = \frac{xi+yj}{4}.$

$n\cdot j = \left(\frac{xi+yj}{4}\right)\cdot j = \frac{y}{4}.$

$F.n = (z^2i + x^2j - y^2zk)\cdot\left(\frac{xi+yj}{4}\right) = \frac{z^2x + x^2y}{4}$

$\iint_S F.n\,ds = \iint_R \frac{z^2x+x^2y}{4}\cdot\frac{dx\,dz}{(y/4)} = \iint\left(\frac{z^2x}{\sqrt{16-x^2}} + x^2\right)dx\,dz$

$x^2+y^2=16$

$y = \sqrt{16-x^2}$

$y=4.$

$= \int_0^4\int_0^5 \left(\frac{z^2x}{\sqrt{16-x^2}} + x^2\right)dx\,dz$

$= \int_0^4 \left[\frac{x}{\sqrt{16-x^2}}\cdot\frac{z^3}{3} + x^2z\right]_0^5 dx = \int_0^4\left[\frac{125}{3}\cdot\frac{x}{\sqrt{16-x^2}} + 5x^2\right]dx$

$= \frac{125}{3}\int_0^4 \frac{x}{\sqrt{16-x^2}}dx + 5\int_0^4 x^2dx$

let $16-x^2 = t$

$-2x\,dx = dt$

$x\,dx = -dt/2$

Limits:

x	0	4
t	16	0

$= \frac{-1}{2}\times\frac{125}{3}\int_{16}^{0}\frac{1}{t^{1/2}}dt + \left(\frac{5x^3}{3}\right)_0^4 = \frac{-125}{6}\left[\frac{t^{1/2}}{1/2}\right]_{16}^{0} + \frac{5\times 64}{3}$

$= \frac{-125}{6}\times 2(-4) + \frac{320}{3} = \frac{500}{3} + \frac{320}{3}$

$= \frac{820}{3}.$

VOLUME INTEGRAL:

Any integral which is to be evaluated, over a volume is called volume integral.

If $\vec{F}(x,y,z)$ is vector point function, then the volume integral is given by $\iiint_V \vec{F}\cdot dv$

Q1: Evaluate $\iiint_V \nabla\cdot\vec{F}\,dv$ where $\vec{F}=(2x^2-4z)i-2xyj-8x^2k$ vector and 'V' is bounded by the plane $x=0, y=0, z=0$ and $x+y+z=1$.

Sol: Given,

$$\vec{F}=(2x^2-4z)i-2xyj-8x^2k$$

$$\phi = x+y+z-1$$

$$\nabla\cdot F = \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\left((2x^2-4z)i-2xyj-8x^2k\right)$$

$$= 4x-2x-0 = 2x.$$

$x+y+z=1$

$z=1-x-y$

$z=0$

$1-x=y$

$y=0$

$x=1$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{z=1-x-y}[2x]\,dx\,dy\,dz$$

$$=2\int_0^1\int_0^{1-x} x[z]_0^{1-x-y}\,dx\,dy = 2\int_0^1\int_0^{1-x} x(1-x-y)\,dx\,dy$$

$$=2\int_0^1\int_0^{1-x}(x-x^2-xy)\,dx\,dy = 2\int_0^1\left[xy-x^2y-\frac{xy^2}{2}\right]_0^{1-x}dx$$

$$=2\int_0^1\left[x(1-x)-x^2(1-x)-\frac{x(1-x)^2}{2}\right]dx$$

$$= 2\int_0^1 \left[x - x^2 - x^2 + x^3 - \frac{x(1+x^2-2x)}{2}\right] dx$$

$$= 2\int_0^1 \left(x - 2x^2 + x^3 - \frac{x}{2} - \frac{x^3}{2} + \frac{2x^2}{2}\right) dx$$

$$= 2\int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx = 2\left[\frac{1}{2}\cdot\frac{x^2}{2} - \frac{x^3}{3} + \frac{1}{2}\cdot\frac{x^4}{4}\right]_0^1$$

$$= 2\left[\frac{1}{4} - \frac{1}{3} + \frac{1}{8}\right] = 2\left(\frac{6-8+3}{24}\right) = 1/12.$$

Q2: Evaluate $\iiint \phi\, dv$ where $\phi = 45x^2y$ and 'v' is a closed region bounded by a plane $4x+3y+z=8$ and $x=y=z=0$

Sol: Given, $\phi = 4x+3yz-8$

$$\iiint_V \phi\, dv = \iiint_V 45x^2y\, dx\, dy\, dz$$

$4x+3y+z-8=0$

$z = 8-4x-3y$

$z=0,\ y=\frac{8-4x}{3}$

$y=0,\ x=2$

$$\int_{x=0}^{x=2}\int_{y=0}^{y=\frac{8-4x}{3}}\int_{z=0}^{8-4x-3y} 45x^2y\, dx\, dy\, dz$$

$$\int_{x=0}^{2}\int_{y=0}^{\frac{8-4x}{3}} \left[45x^2yz\right]_0^{8-4x-3y} dx\, dy = 45\int_{x=0}^{2}\int_{y=0}^{\frac{8-4x}{3}} x^2y\,(8-4x-3y)\, d$$

$$= 45\int_0^2\int_0^{\frac{8-4x}{3}} (8x^2y - 4x^3y - 3x^2y^2)\, dx\, dy$$

$$= 45\int_0^2 \left[\frac{8x^2y^2}{2} - \frac{4x^3y^2}{2} - \frac{3x^2y^3}{3}\right]_0^{\frac{8-4x}{3}} dx$$

$$= 45\int_0^2 \left(4x^2y^2 - 2x^3y^2 - x^2y^3\right)_0^{\frac{8-4x}{3}} dx$$

$$= 45\int_0^2 \left(4x^2\frac{(8-4x)^2}{9} - 2x^3\frac{(8-4x)^2}{9} - x^2\frac{(8-4x)^3}{27}\right) dx$$

$$= \frac{45}{9}\int_0^2 x^2\left[4(8-4x)^2 - 2x(8-4x)^2 - \frac{(8-4x)^3}{3}\right] dx$$

$$= 5\int_0^2 x^2\left[4(64+16x^2-64x) - 2x(64+16x^2-64x) - \left(\frac{-64x^3+384x^2-768x+512}{3}\right)\right] dx$$

$$= 5\int_0^2 x^2\left[256+64x^2-256x-128x-32x^3+128x^2+\frac{64x^3-384x^2+768x-512}{3}\right] dx$$

$$= \frac{5}{3}\int_0^2 x^2\left[768+192x^2-768x-384x-96x^3+384x^2+64x^3-384x^2+768x-512\right] dx$$

$$= \frac{5}{3}\int_0^2 x^2\left[352x^3-192x^2-384x+256\right] dx$$

$$= \frac{5\times32}{3}\int_0^2 \left(11x^5-6x^4-12x^3+8x^2\right) dx$$

$$= \frac{5\times32}{3}\left[\frac{11x^6}{6} - \frac{6x^5}{5} - \frac{12x^4}{4} + \frac{8x^3}{3}\right]_0^2 = \frac{5\times32}{3}\left[\frac{11\times64}{6} - \frac{6\times32}{5} - \frac{12\times16}{4} + \frac{8\times8}{3}\right]$$

$$= \frac{160}{3}\left[\frac{704}{6} - \frac{192}{5} - 48 + \frac{64}{3}\right] = \frac{160}{3}\times\frac{3520-1152-1440+640}{30}$$

Q3. If $\vec{F} = (2x^2-3z)i - 2xyj - 4xk$, where 'V' is bounded by plane $x=0, y=0, z=0$ and $2x+2y+z=4$. $\iiint \nabla\cdot F\, dv$

Sol: Given, $\vec{F} = (2x^2-3z)i - 2xyj - 4xk$

$$\phi = 2x+2y+z-4$$

$$\nabla\cdot F = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left[(2x^2-3z)i - 2xyj - 4xk\right]$$

$$= \frac{\partial}{\partial x}(2x^2-3z) - \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x + 0 = 2x.$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\, dx\, dy\, dz$$

$2x+2y+z-4=$
$z = 4-2x-2y$
$z=0,$
$y = 2-x$
$y=0$
$x=2.$

$$= 2\int_0^2\int_0^{2-x} x(z)_0^{4-2x-2y}\, dx\, dy = 2\int_0^2\int_0^{2-x} x(4-2x-2y)\, dx\, dy$$

$$= 2\int_0^2\int_0^{2-x}(4x-2x^2-2xy)\, dx\, dy = 2\int_0^2\left(4xy - 2x^2y - \frac{2xy^2}{2}\right)_0$$

$$= 2\int_0^2\left(4x(2-x) - 2x^2(2-x) - x(2-x)^2\right)dx$$

$$= 2\int_0^2\left(8x - 4x^2 - 4x^2 + 2x^3 - 4x - x^3 + 4x^2\right)dx$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\,dx = 2\left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right)_0^2$$

$$= 2\left[\frac{16}{4} - \frac{4}{3}\cdot 8 + 2(4)\right] = 2\left[4 - \frac{32}{3} + 8\right]$$

$$= 2\left(12 - \frac{32}{3}\right) = 2\times\frac{4}{3} = \frac{8}{3}.$$

Q4: Evaluate $\iiint_V \nabla\cdot F\,dv$ where $\vec{F} = x^2 i + y^2 j + z^2 k$ and V is the volume enclosed by cube $0 \le x, y, z \le 1$.

Sol: Given, $\vec{F} = x^2 i + y^2 j + z^2 k$

$$\nabla\cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2 i + y^2 j + z^2 k)$$

$$= 2x + 2y + 2z = 2(x+y+z).$$

$$\iiint \nabla\cdot F\,dv = \iiint 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz = 2\int_0^1\int_0^1 \left(xz + yz + \frac{z^2}{2}\right)_0^1 dx\,dy$$

$$= 2\int_0^1\int_0^1 \left(x + y + \frac{1}{2}\right)dx\,dy = 2\int_0^1 \left(xy + \frac{y^2}{2} + \frac{y}{2}\right)_0^1 dx$$

$$= 2\int_0^1 \left(x + \frac{1}{2} + \frac{1}{2}\right)dx = 2\int_0^1 (x+1)\,dx = 2\left[\frac{x^2}{2} + x\right]_0^1$$

$$= 2\left(\frac{1}{2}+1\right) = \frac{3}{2}\times 2 = 3.$$

Q5: Evaluate $\iiint A\,dv$ where $A = 2xz\hat{i} - x\hat{j} + y^2\hat{k}$, where 'v' is bounded by surface $x=0, y=0, z=x^2, z=4$ and $x=2, y=6$

Sol:

$$\iiint A\,dv = \iiint (2xz\hat{i} - x\hat{j} + y^2\hat{k})\,dv$$

$$= \hat{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dx\,dy\,dz - \hat{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dx\,dy\,dz + k\int_0^2\int_0^6\int_{x^2}^4 y^2\,dx\,dy\,dz$$

$$= \hat{i}\int_0^2\int_0^6 \left(\frac{2x z^2}{2}\right)_{x^2}^4 dx\,dy - \hat{j}\int_0^2\int_0^6 x(z)_{x^2}^4\,dx\,dy + k\int_0^2\int_0^6 y^2(z)_{x^2}^4\,dx\,dy$$

$$= \hat{i}\int_0^2\int_0^6 (16x - x^5)\,dx\,dy - \hat{j}\int_0^2\int_0^6 (4x - x^3)\,dx\,dy + k\int_0^2\int_0^6 (4y^2 - x^2y^2)\,dx\,dy$$

$$= \left[\hat{i}\int_0^2 (16xy - x^5y)_0^6 - \hat{j}\int_0^2 (4xy - x^3y)_0^6 + k\int_0^2 \left(\frac{4y^3}{3} - \frac{x^2y^3}{3}\right)_0^6\right] dx\,dy$$

$$= \hat{i}\int_0^2 (96x - 6x^5)\,dx - \hat{j}\int_0^2 (24x - 6x^3)\,dx + \frac{k}{3}\int_0^2 (864 - 216x^2)$$

$$= \hat{i}\left[\frac{96x^2}{2} - \frac{6x^6}{6}\right]_0^2 - \hat{j}\left[\frac{24x^2}{2} - \frac{6x^4}{4}\right]_0^2 + \frac{k}{3}\left[864x - \frac{216x^3}{3}\right]_0^2$$

$$= \hat{i}\left(48(4) - \frac{8(64)}{8}\right) - \hat{j}\left(12(4) - \frac{3(16)}{2}\right) + \frac{k}{3}\left((864\times 2) - \frac{1728}{3}\right)$$

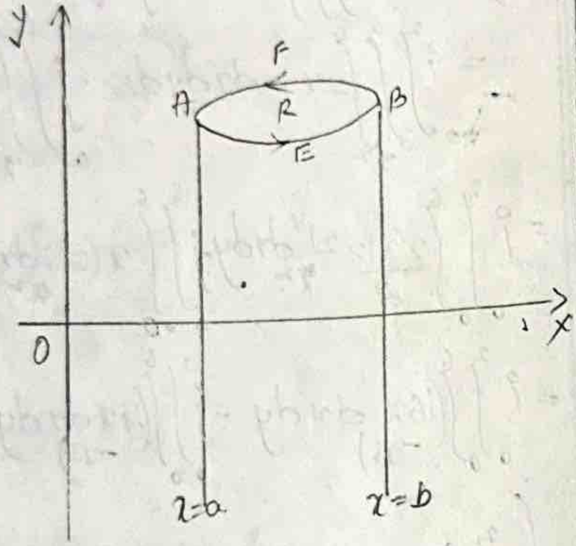
$$= \frac{\cancel{(576-512)}}{\cancel{3}}\hat{i} - (48-24)\hat{j} + \frac{k}{3}(1728-576)$$

$$= (192-64)\hat{i} - (24\hat{j}) + k\left(\frac{1152}{3}\right) = 128\hat{i} - 24\hat{j} + 384k$$

$$\iiint F\,dV = 128\hat{i} - 24\hat{j} + 384k.$$

GREEN'S, GAUSS DIVERGENCE & STOKE'S THEOREM AND APPLICATIONS

GREEN'S THEOREM:



Statement:

If 'R' is a closed region in xy-plane bounded by a simple closed curve and M and N be continuous of 'x' and 'y' having continuous derivatives in R, then

$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

where 'c' travels in +ve direction.

Q1: Apply Green's Theorem to evaluate $\oint_C (2xy-y^2)dx+(x^2+y^2)dy$, where 'C' is a closed curve bounded by $y=x^2$ and $y^2=x$.

Sol: Given, $\oint_C (2xy-y^2)dx+(x^2+y^2)dy$

By Green's theorem, $\oint_C Mdx+Ndy = \iint_R \left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right] dx\, dy$

Where $M = 2xy-y^2$, $N = x^2+y^2$

$\frac{\partial M}{\partial y} = 2x-2y$ $\quad \frac{\partial N}{\partial x} = 2x$

$\oint (2xy-y^2)dx+(x^2+y^2)dy = \iint (2x-2x+2y)dx\,dy$

$= \iint 2y\, dx\, dy$

$= \int_0^1 \int_{\sqrt{x}}^{x^2} 2y\, dx\, dy = \int_0^1 \left[\cancel{2}\frac{y^2}{\cancel{2}}\right]_{\sqrt{x}}^{x^2}$

$= \int_0^1 (y^2)_{\sqrt{x}}^{x^2} dx = \int_0^1 (x^4 - x)\, dx$

$= \left(\frac{x^5}{5}\right)_0^1 - \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{5} - \frac{1}{2} = \frac{2-5}{10} = \frac{-3}{10} = \frac{3}{10}.$

$y = x^2;\ y^2 = x$

$y = \sqrt{x}$

$x^2 = \sqrt{x}$.

$x^2 - \sqrt{x} = 0$

$\sqrt{x}(x^{3/2}-1) = 0$

$x = 1, x = 0$

Q2: Apply Green's Theorem to evaluate $\oint_C x^2y\,dx + y^3 dy$. where 'C' is a closed path formed by $y=x$ and $y=x^3$ from (0,0) to (1,1).

Sol: Given, $\oint_C x^2y\,dx + y^3dy$

By Green's Theorem, $\oint_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = x^2y$ $\qquad N = y^3$

$\partial M/\partial y = x^2$ $\qquad \partial N/\partial x = $ $\qquad (x^2)^2$

$$\oint_C x^2y\,dx + y^3dy = \int_0^1\int_x^{x^3}(-x^2)\,dx\,dy$$

$y=x,\ y=x^3$

$x-x^3=0$

$x(1-x^2)=0$

$x=0, x=1$

$$= \int_0^1 -x^2(y)_x^{x^3}dx = -\int_0^1 (x^2)(x^3-x)\,dx$$

$$= -\int_0^1 (x^5-x^3)\,dx = \int_0^1 (x^3-x^5)\,dx = \left(\frac{x^4}{4}-\frac{x^6}{6}\right)_0^1$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Q3: Evaluate $\oint_C (xy^2+2xy)\,dx + x^2y\,dy$ where 'c' is the boundary of the region enclosing $y^2=4x$ and $x=3$.

Sol: Given, $\oint_C (xy^2+2xy)\,dx + x^2y\,dy$.

By Green's Theorem, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy$

$M = xy^2+2xy$ $\qquad$ $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy+2x$ $\qquad$ $\frac{\partial N}{\partial x} = 2xy$

$$\oint_C (xy^2+2xy)\,dx + x^2y\,dy = \iint_R (2xy-2xy-2x)\,dx\,dy = \iint_R (-2x)\,dx\,dy$$

$$= \int_0^3\int_{y=-4x}^{y=4x} (-2x)\,dx\,dy = \int_0^3 (-2xy)_{-4x}^{4x}\,dx = \int_0^3 (-8x^2-8x^2)\,dx$$

$$= \int_0^3 -16x^2\,dx = -16\left(\frac{x^3}{3}\right)_0^3 = \frac{-16\times 27}{3} = -144.$$

Q4: Evaluate by Green's Theorem $\oint_C e^{-x}[\sin y\,dx + \cos y\,dy]$ where 'c' is rectangular vertices enclosing $(0,0)$, $(\pi,0)$, $(\pi,\pi/2)$, $(0,\pi/2)$.

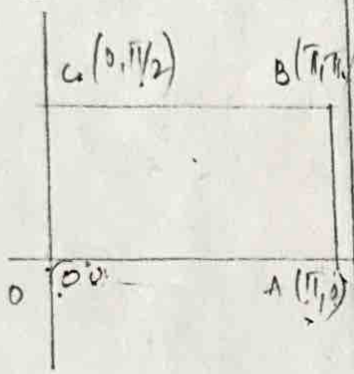
Sol: Given, $\oint_C e^{-x}[\sin y\,dx + \cos y\,dy]$.

By Greens Theorem, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = e^{-x}\sin y \qquad N = +e^{-x}\cos y$

$\frac{\partial N}{\partial x} = -e^{-x}\cos y$

$\frac{\partial M}{\partial y} = e^{-x}\cos y$



$$\oint_C e^{-x}[\sin y\,dx + \cos y\,dy] = \int_0^{\pi}\int_0^{\pi/2} -2e^{-x}\cos y\,dx\,dy$$

$$= -2\int_0^{\pi} e^{-x}(\sin y)_0^{\pi/2}\,dx = -2\int_0^{\pi} e^{-x}\,dx$$

$$= -2\left[\frac{e^{-x}}{-1}\right]_0^{\pi} = +2\left[e^{-\pi} - 1\right]$$

$$= 2\left(e^{-\pi} - 1\right).$$

Q5: Evaluate by Greens Theorem $\oint_C (\cos x \sin y - xy)\,dx + (\sin x \cos y)\,dy$, where 'c' is a circle $x^2 + y^2 = 1$.

Sol: Given, $\oint_C (\cos x \sin y - xy)dx + (\sin x \cos y)dy$

By Green's Theorem, $\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = \cos x \sin y - xy$ $\qquad$ $N = \sin x \cos y$

$\frac{\partial M}{\partial y} = \cos x \cos y - x$ $\qquad$ $\frac{\partial N}{\partial x} = \cos x \cos y$

$\oint (\cos x \sin y - xy)\,dx + (\sin x \cos y)\,dy = \iint_R (\cos x \cos y - \cos x \cos y + x)\,dx\,dy$

$= \iint_R x \, dx\,dy$

$x^2 + y^2 = 1$

$x = 0, y = 1$

$y = \sqrt{1-x^2}$

$= \int_0^1 \int_0^{\sqrt{1-x^2}} x \, dx\,dy = \int_0^1 x (y)_0^{\sqrt{1-x^2}} \, dx\,dy$

$= \int_0^1 x(\sqrt{1-x^2})\,dx$

put $x = \sin\theta$ $\qquad$ Limits: x : 1, 0

$dx = \cos\theta \, d\theta$ $\qquad$ $\sin\theta$: $\pi/2$, 0

$\int_0^{\pi/2} \sin\theta \cdot \cos\theta \cdot \cos\theta \, d\theta = \int_0^{\pi/2} \sin\theta \cos^2\theta \, d\theta$

Formula: If $m = 1$, then $\int_0^{\pi/2} \sin x \cos^n x \, dx = \frac{1}{n+1}$.

$= \frac{1}{2+1} = \frac{1}{3}$.

$\int e^{ax} dx = \frac{e^{ax}}{a}$

(4)

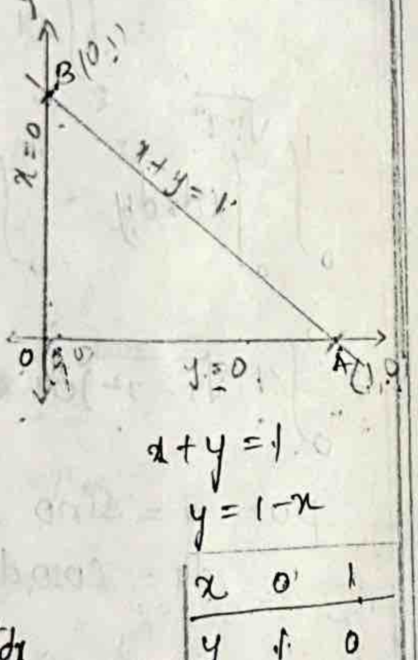
Q6: Verify Green's Theorem for $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$ where 'c' is the region bounded by triangle $x=0, y=0$, $x+y=1$.

Sol: Given, $\int_C (3x^2-8y^2)dx+(4y-6xy)dy$

By Green's Theorem, $\int_C Mdx+Ndy = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

$\int_C (3x^2-8y^2)dx+(4y-6yx)dy =$

$\iint \left(\frac{\partial}{\partial x}(4y-6xy)-\frac{\partial}{\partial y}(3x^2-8y^2)\right)dx\,dy.$



$L.H.S = \int_C (3x^2-8y^2)dx+(4y-6xy)dy$

Along OA: $y=0$ $x:(0\to 1)$
$dy=0$

$x+y=1$
$y=1-x$

x	0	1
y	1	0

$\int_{OA}(3x^2-8y^2)dx+(4y-6yx)dy = \int_0^1 3x^2dx$

$= \frac{3(x^3)_0^1}{3} = 1$

Along AB: $y=1-x \Rightarrow x=1-y$ $y:(0\to 1)$
$dy=-dx$ $dx=-dy$

$\int_{AB}(3x^2-8y^2)dx+(4y-6xy)dy = \int_0^1 (3(1-y)^2-8y^2)(-dy)+4y-6y(1-y)dy$

$= \int_0^1 [-3(1-y^2)+8y^2+4y-6y(1-y)]dy$

$$= \int_0^1 -(3(1+y^2-2y) - 8y^2)\,dy + (4y - 6y + 6y^2)\,dy$$

$$= \int_0^1 (-3y^2 + 6y - 3 + 4y - 6y + \underset{8y^2}{\cancel{6y^2}})\,dy$$

$$= \int_0^1 (11y^2 + 4y - 3)\,dy = \frac{11}{3}(y^3)_0^1 + \frac{4}{2}(y^2)_0^1 - 3(y)_0^1$$

$$= \frac{11}{3} + 2 - 3 = \frac{11}{3} - 1 = \frac{8}{3}.$$

Along OB $x = 0$ $y : (1 \to 0)$
$dx = 0$

$$\int_{OB} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_1^0 4y\,dy = 2(y^2)_1^0$$
$$= -2.$$

$$\therefore L.H.S = \int_{OAB} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \frac{8}{3} + 1 - 2 = \frac{5}{3}.$$

$$R.H.S: \iint_R \left(\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right) dx\,dy.$$

$$= \iint (-6y + 16y)\,dx\,dy = \int_0^1 \int_0^{1-x} 10y\,dx\,dy = \int_0^1 \frac{10(y^2)_0^{1-x}}{2}\,dx$$

$$= 5\int_0^1 (1-x)^2\,dx = 5\int_0^1 1 + x^2 - 2x = 5\left[x + \frac{x^3}{3} - \frac{2x^2}{2}\right]_0^1$$

$= 5\left[1 + \frac{1}{3} - 1\right] = \frac{5}{3}$.

L.H.S = R.H.S

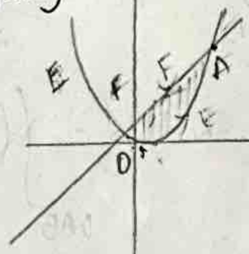
$\therefore$ Green's Theorem is Verified.

Q7: Verify Green's Theorem for plane $\oint_C (xy+y^2)\,dx + x^2 dy$ where 'C' is the closed curve for the region bounded by $y=x$ and $y=x^2$.

Sol: Given, $\oint_C (xy+y^2)\,dx + x^2 dy$

By Green's theorem, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\oint_C (xy+y^2)\,dx + x^2 dy = \iint_R \left[\frac{\partial}{\partial x} x^2 - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy$.



Along OEA: $y = x^2$ $\quad x : (0 \to 1)$

$dy = 2x\,dx$

$$\int_{OEA} (xy+y^2)\,dx + x^2 dy = \int_0^1 (x \cdot x^2 + x^4)\,dx + (x^2 \cdot 2x)\,dx$$

$$= \int_0^1 (x^3 + x^4 + 2x^3)\,dx = \int_0^1 (3x^3 + x^4)\,dx = \left(\frac{3x^4}{4} + \frac{x^5}{5}\right)_0^1$$

$$= \frac{19}{20}.$$

Along AFO: $y = x$ $\quad x:(1 \to 0)$

$dy = dx$

$$\int_{AFO} (xy+y^2)dx + x^2dy = \int_1^0 (x \cdot x + x^2 + x^2)dx = \int_1^0 3x^2 dx = \left(\frac{3x^3}{3}\right)$$

$$= (x^3)_1^0 = -1.$$

$$L.H.S = \int (xy+y^2)dx + x^2dy = \frac{19}{20} - 1 = \frac{1}{20}$$

$y = x$, $y = x^2$

$x - x^2 = 0$

$x = 0, x = 1$

$$R.H.S = \iint \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right) dx\,dy$$

$$= \int_0^1 \int_x^{x^2} (2x - x - 2y)\,dx\,dy = \int_0^1 \left(xy - \frac{2y^2}{2}\right)_x^{x^2} dx$$

$$= \int_0^1 \left((x^3 - x^2) - (x^4 - x^2)\right) dx = \int_0^1 (x^3 - \cancel{x^2} - x^4 + \cancel{x^2})\,dx$$

$$= \int_0^1 (x^3 - x^4)\,dx = \left(\frac{x^4}{4} - \frac{x^5}{5}\right)_0^1 = \frac{1}{4} - \frac{1}{5} = \frac{1}{20}.$$

$\therefore L.H.S = R.H.S$

$\therefore$ Green's Theorem is Verified.

Q8: $\int_C (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$, Verify Green's Theorem where 'c' is the boundary region bounded $y^2 = 8x$, $x = 2$.

Sol: Given, $\int_C (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$

By Green's Theorem,

$$\int M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$

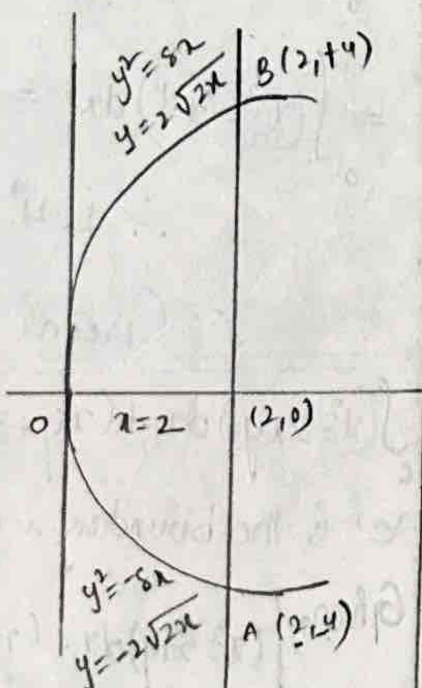
$$\int (x^2-2xy)dx + (x^2y+3)dy = \iint_R \left[\frac{\partial}{\partial x}(x^2y+3) - \frac{\partial}{\partial y}(x^2-2xy)\right] dx\, dy$$

$$\text{R.H.S} = \int_0^2 \int_{-2\sqrt{2x}}^{2\sqrt{2x}} (2xy+2x)\, dx\, dy = \int_0^2 \left(\frac{2xy^2}{2} + 2xy\right)_{-2\sqrt{2x}}^{2\sqrt{2x}} dx$$

$$= \int_0^2 \left[x(2\sqrt{2x})^2 + 2x(2\sqrt{2x})\right] - \left[x(-2\sqrt{2x})^2 + 2x(-2\sqrt{2x})\right] dx$$

$$= \int_0^2 8x^2 + 4\sqrt{2}x^{3/2} - 8x^2 + 4\sqrt{2}x^{3/2} = \int_0^2 8\sqrt{2}x^{3/2}$$

$$= 8\sqrt{2}\,\frac{\left(x^{5/2}\right)_0^2}{5/2} = \frac{8\times\sqrt{2}\times 2\times\sqrt{25}}{5} = \frac{128}{5}.$$



Along AO : BO :–

$y = 2\sqrt{2}\sqrt{x}$

$dy = \cancel{2}\sqrt{2} \cdot \frac{1}{\cancel{2}\sqrt{x}} dx = \sqrt{2}\, x^{-\frac{1}{2}} dx$

$$\int (x^2 - 2xy)dx + (x^2 y + 3)dy = \int_2^0 x^2 - 2x(2\sqrt{2}\sqrt{x})dx + (x^2(1)\sqrt{2x} + 3)(\sqrt{2}\, x^{-1/2} dx)$$

$$= \int x^2 - 4\sqrt{2}\, x^{3/2} + 4x^2 + 3(2x^{-1/2})dx$$

$$= -\int_0^2 \left(\frac{5x^3}{3} - 4\sqrt{2}\frac{x^{5/2}}{5/2} + 3\sqrt{2}\frac{x^{1/2}}{1/2} \right]_0^2$$

$$= -\left[5 \times \frac{8}{3} - \frac{4 \times 2\sqrt{2} \times 4\sqrt{2}}{5} + 6(2)\right]$$

$$= \frac{40}{3} - \frac{64}{5} + 12 = -\frac{188}{15}$$

Along AB:

$x = 2$

$dx = 0 \quad y = -4 \to 4$

$$\int (x^2 - 2xy)dx + (x^2 y + 3)dy = \int_{-4}^{4} (4y + 3)dy$$

$$= \left(4\frac{y^2}{2} + 3y\right)_{-4}^{4} = 24$$

$$L.H.S = \int_{OAB} (x^2 - 2xy)dx + (x^2 y + 3)dy$$

$$= \frac{212}{15} - \frac{188}{15} + 24 = \frac{128}{5}$$

$\therefore$ LHS = RHS

$\therefore$ Green's theorem is verified

$L.H.S = \iint (x^2 - 2xy)\,dx + (x^2y + 3)\,dy$

Along OB: $y = -2\sqrt{2}\sqrt{x}$

$dy = -2\sqrt{2} \times \frac{1}{2\sqrt{x}}\,dx$ $x = 0 \to 2$

$= \int_0^2 (x^2 - 2x(-2\sqrt{2}\sqrt{x}))\,dx + (x^2(-2\sqrt{2}\sqrt{x}) + 3)(-\sqrt{2}x^{-1/2})\,dx$

$= \int_0^2 (x^2 + 4\sqrt{2}\,x^{3/2} + 4x^2 - 3\sqrt{2}\,x^{-1/2}]\,dx$

$= \int_0^2 (5x^2 + 4\sqrt{2}\,x^{3/2} - 3\sqrt{2}\,x^{-1/2})\,dx$

$= \left[\frac{5x^3}{3} + 4\sqrt{2}\cdot\frac{2}{5}x^{5/2} - 3\sqrt{2}(2)x^{-1/2}\right]_0^2$

$= \left[\frac{5x^3}{3} + \frac{8}{5}\sqrt{2}\,x^{5/2} - 6\sqrt{2}\frac{x^{-1/2}}{1/2}\right]_0^2$

$= \left[\frac{5(2)^3}{3} + \frac{8}{5}\sqrt{2}\,x^{5/2} - 12\sqrt{2}\,x^{-1/2}\right]$

$= \frac{40}{3} + \frac{64}{5} - 12 \quad \cancel{\frac{40}{3}}$

$= \frac{40}{3} + \frac{64-60}{5}$

$= \frac{40}{3} + \frac{4}{5} = \frac{200+12}{15} = \frac{212}{15}$

Q9: Verify Green's theorem in plane $\oint_C (x^2 - xy^3$ … where 'C' is the square with the vertices … $(2,2)$ $(0,2)$.

Sol: By Green's Theorem,

$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy =$

$\iint_R \left[\frac{\partial}{\partial x}(y^2 - 2xy) - \frac{\partial}{\partial y}(x^2 - xy^3)\right] dx\,dy$

L.H.S: $\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ —

Along OA: $y = 0$ $x : (0 \to 2)$

$dy = 0$

$\int_{OA} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_0^2 (x^2 -$ …

$= \int_0^2 x^2\,dx = \left(\frac{x^3}{3}\right)_0^2 =$ …

Along AB: $x = 2$ $y : (0 \to 2)$

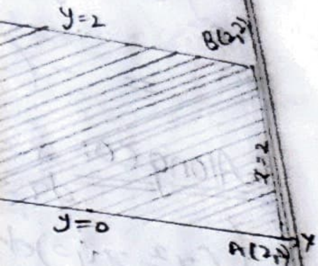
$dx = 0$

$\int_{AB} (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy = \int_0^2 (y^2$ …

$= \left(\frac{y^3}{3} - \frac{4y^2}{2}\right)_0^2 = \frac{8}{3} - 2(4)$ …

$) dx + (y^2 - 2xy) dy$

$(0,0) \quad (2,0)$



$\rightarrow$ (A)

$xy^3) dx + 0$

$8/3.$

$-4y) dy$

$= \frac{8}{3} - 8.$

Along BC: $y=2$ $\quad x:(2\to 0)$
$dy=0$

$$\int_{BC}(x^2-xy^3)\,dx+0=\int_2^0(x^2-8x)\,dx=\left(\frac{x^3}{3}\right)_2^0-8\left(\frac{x^2}{2}\right)_2^0$$

$$=-\left(\frac{8}{3}-4(4)\right)=16-\frac{8}{3}.$$

Along CO: $x=0$ $\quad y:(2\to 0)$
$dx=0$

$$\int_{CO}(x^2-xy^3)\,dx+(y^2-2xy)\,dy=\int_2^0 y^2\,dy=\left(\frac{y^3}{3}\right)_2^0=\frac{-8}{3}.$$

Along OABC: $\displaystyle\int_{OABC}(x^2-xy^3)\,dx+(y^2-2xy)\,dy=\frac{8}{3}+\frac{8}{3}-8+16-\frac{8}{3}-\frac{8}{3}$

$$=\frac{16}{3}-\frac{16}{3}+8=8.$$

R.H.S: $\displaystyle\iint_R\left(\frac{\partial}{\partial x}(y^2-2xy)-\frac{\partial}{\partial y}(x^2-xy^3)\right)dx\,dy$

$$=\iint_R(-2y+3y^2x)\,dx\,dy=\int_0^2\int_0^2(3xy^2-2y)\,dx\,dy$$

$$=\int_0^2\left(\frac{3xy^3}{3}-\frac{2y^2}{2}\right)_0^2dx=\int_0^2(8x-4)\,dx=\left(\frac{8x^2}{2}-4x\right)_0^2$$

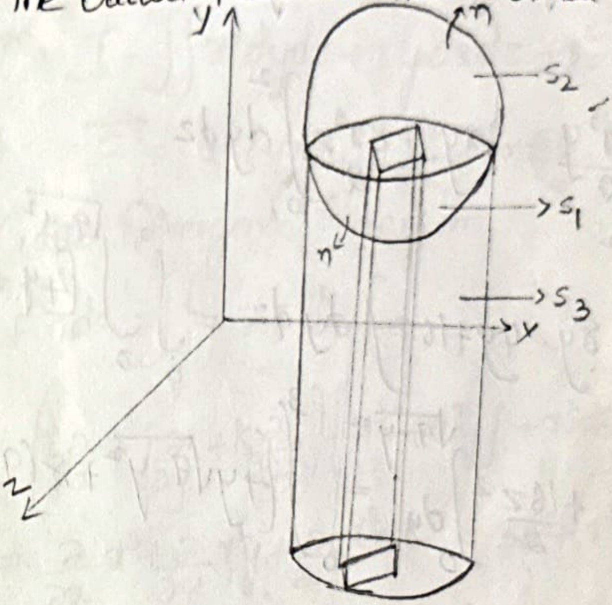
$$=4(4)-4(2)=16-8=8.$$

$\therefore$ L.H.S = R.H.S

$\therefore$ Hence Green's Theorem is Verified.

GUASS DIVERGENCE THEOREM –

STATEMENT: If $\vec{F}$ is a vector point function having continuous first order partial derivative a closed surface enclosing a volume V, then $\iint_S F.n\,ds = \iiint_V \nabla.F\,dv$, where 'n' is the outward drawn unit normal to the surface.



Q1: Evaluate $\iint_S F.n\,ds$ where $\vec{F} = 2x^2yi - y^2j + 4xz^2k$ where 'S' is the surface $y^2+z^2=9$ and $x=2$, in the first octant.

Sol: Given, $\vec{F} = 2x^2yi - y^2j + 4xz^2k$

By Gauss Divergence Theorem,

$$\iint_S F.n\,ds = \iiint_V \nabla.F\,dv$$

$$\nabla.F = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right]\left(2x^2yi - y^2j + 4xz^2k\right)$$

$$= 4xy - 2y + 8xz$$

$$\iint_S F\cdot n\,ds = \iiint_V (4xy-2y+8xz)\,dx\,dy\,dz$$

$y^2+z^2=9$

$z^2=9-y^2$

$z=\sqrt{9-y^2}$

$$= \int_0^2\int_0^3\int_0^{\sqrt{9-y^2}} (4xy-2y+8xz)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^{\sqrt{9-y^2}} \left[\frac{4x^2y}{2} - 2xy + \frac{8x^2z}{2}\right]_0^2 dy\,dz$$

$$= \int_0^3\int_0^{\sqrt{9-y^2}} \left[8y-4y+16z\right] dy\,dz = \int_0^3\int_0^{\sqrt{9-y^2}} \left[4y+16z\right] dy\,dz$$

$$= \int_0^3 \left[4yz + \frac{16z^2}{2}\right]_0^{\sqrt{9-y^2}} dy = \int_0^3 \left[4y\sqrt{9-y^2} + 8(9-y^2)\right] dy$$

Put $9-y^2=t^2$

$-2y\,dy = 2t\,dt$

$y\,dy = -dt\cdot t$

Limits: y 0 3

$t=\sqrt{9-y^2}$ 3 0

$$= \int_0^3 4y\sqrt{9-y^2}\,dy + 8\int_0^3 (9-y^2)\,dy$$

$$= \int_3^0 4\cdot t(-t)\,dt + 8\left[\int_0^3 9\,dy - \int_0^3 y^2\,dy\right]$$

$$= -4\int_3^0 t^2\,dt + 8\left[(9y)_0^3 - \left(\frac{y^3}{3}\right)_0^3\right]$$

$$= -4\left(\frac{t^3}{3}\right)_3^0 + 8\left[9(3) - \frac{27}{3}\right] = -4(0-9) + 8(27-9)$$

$$= 36 + (8\times 18) = 36+144$$

$$= 180.$$

Q2: Apply Gauss-Divergence theorem to evaluate $I = \iint x^3 dy dz + x^2 y dz dx + x^2 z dx dy$, where 's' is the closed surface bounded by the planes $z=0$, $z=b$ and the cylinder $x^2+y^2=a^2$.

Sol: Given, $I = \iint x^3 dy dz + x^2 y dx dz + x^2 z dx dy$

$$\vec{F} = x^3 i + x^2 y j + x^2 z k$$

By Gauss Divergence theorem,

$$\iint_S F \cdot n ds = \iiint_V \nabla \cdot F dV$$

$$\nabla \cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(x^3 i + x^2 y j + x^2 z k\right)$$

$$= \frac{\partial}{\partial x} x^3 + \frac{\partial}{\partial y} x^2 y + \frac{\partial}{\partial z} x^2 z = 3x^2 + x^2 + x^2 = 5x^2.$$

$$I = \iint_S x^3 dy dz + x^2 y dz dx + x^2 z dx dy = \iiint 5x^2 dx dy dz \longrightarrow (A)$$

$$= \int_{z=0}^{b} \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} 5x^2 dx dy dz = \int_{z=0}^{b} \int_{y=0}^{a} \left(\frac{5x^3}{3}\right)_0^{\sqrt{a^2-y^2}} dy dx.$$

$$= \int_{z=0}^{b} \int_{y=0}^{a} \frac{5}{3} (a^2-y^2)^{3/2} dy dz = \int_{y=0}^{a} \frac{5}{3} (a^2-y^2) [z]_0^b dy$$

$$= \frac{5b}{3} \int_{y=0}^{a} (a^2-y^2)^{3/2} dy.$$

Put $y = a\sin t$
$dy = a\cos t\,dt$

Limits:

y	0	a
$t = \sin^{-1}(y/a)$	0	$\pi/2$

limits $t = \sin^{-1}(y/a)$

l. limit $= \sin^{-1}(\frac{0}{a}) = 0$

∴ u. limit $= \sin^{-1}(\frac{a}{a}) = \pi/2$

$$= \frac{5b}{3}\int_0^{\pi/2}\sqrt{(a^2-a^2\sin^2 t)^3}\cdot a\cos t\,dt$$

$$= \frac{5b}{3}\int_0^{\pi/2} a^3 \cdot a\cos^3 t\cdot \cos t\,dt$$

$$= \frac{5b}{3}a^4\int_0^{\pi/2}\cos^4 t\,dt = \frac{5b}{3}\times a^4\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = \frac{5a^4b\pi}{16}.$$

Q3. If $\vec{F} = 2xyi + yz^2j + xzk$ and 'S' is a rectangular parallelopiped bounded by $x=0, y=0, z=0, x=2, y=1, z=3$. Evaluate $\iint_S F\cdot n\,ds$.

Sol: Given,

$$\vec{F} = 2xyi + yz^2j + xzk$$

According to Gauss-Divergence theorem,

$$\iint_S F\cdot n\,ds = \iiint_V \nabla\cdot F\,dv$$

$$\nabla\cdot F = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left(2xyi + yz^2j + xzk\right)$$

$$= 2y + z^2 + x.$$

$$\iiint_V 2xyi + yz^2j + xzk = \iiint_V (x + 2y + z^2)\,dv$$

$$= \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{3} (x+2y+z^2)\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{1} (3x+6y+9)\,dx\,dy$$

$$= \int_{x=0}^{2} \left(3xy + \frac{6y^2}{2} + 9y\right)_0^1 dx = \int_{x=0}^{2} (3x+3+9)\,dx$$

$$= \left(\frac{3x^2}{2} + 12x\right)_0^2 = \frac{3}{2}(4) + 12(2) = 30.$$

Q4: Verify Gauss-~~Jordan~~ Divergence theorem for $\vec{F} = (x^3 - yz)i - 2x^2yj + zk$ taken over the surface of the cube. $0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a$

Sol: Given. $\vec{F} = (x^3 - yz)i - 2x^2yj + zk$

By Gauss-Divergence Theorem, $\iint_S F \cdot n\,ds = \iiint_V \nabla \cdot F\,dv$

$$\iint_S \left((x^3-yz)i - 2x^2yj + zk\right)\cdot n\,ds = \iiint_V \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left((x^3-yz)i - 2x^2yj + zk\right)$$

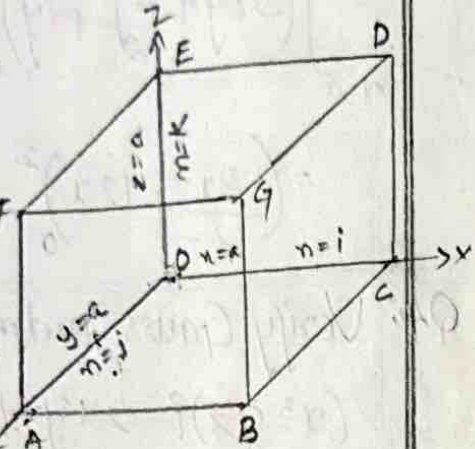
$$R.H.S = \int_0^a\int_0^a\int_0^a \left[\frac{\partial}{\partial x}(x^3-yz) - \frac{\partial}{\partial y}(2x^2y) + \frac{\partial}{\partial z}(z)\right] dv$$

$$\int_0^a\int_0^a\int_0^a (3x^2 - 2x^2 + 1)\,dx\,dy\,dz = \int_0^a\int_0^a\int_0^a (x^2+1)\,dx\,dy\,dz$$

$$\int_0^a\int_0^a (x^2+1)\,dx\,dy\,(z)_0^a = a\int_0^a (x^2+1)\,(y)_0^a\,dx$$

$$= a^2\int_0^a (x^2+1)\,dx = a^2\left(\frac{x^3}{3}+x\right)_0^a = a^2\left(\frac{a^3}{3}+a\right)$$

$$= \frac{a^5}{3}+a^3 \longrightarrow (A).$$



L.H.S: $\iint\left((x^3-yz)i-2x^2yj+zk\right)\cdot n\,ds$

→ Surface BCDG: $x=a, n=i, ds=dydz$

$$\iint_S\left[(x^3-yz)i-2x^2yj+zk\right]\cdot n\,ds$$

$$= \int_0^a\int_0^a\left[(a^3-yz)i-2a^2yj+zk\right]\cdot i\,dy\,dz = \int_0^a\int_0^a (a^3-yz)\,dy\,dz$$

$$= \int_0^a\left(a^3z-\frac{yz^2}{2}\right)_0^a dy = \int_0^a\left(a^4-\frac{ya^2}{2}\right)dy = \left(a^4y-\frac{y^2}{2}\cdot\frac{a^2}{2}\right)_0^a$$

$$= a^5-\frac{a^4}{4} \longrightarrow (1)$$

→ SURFACE OAEF: $x=0, n=-i, ds=dydz$.

$$\iint\left((x^3-yz)i-2x^2yj+zk\right)\cdot n\,ds = \iint\left((0-yz)i-0\cdot j+z\cdot k\right)(-i)\,ds$$

$$= \int_0^a\int_0^a yz\,dy\,dz = \int_0^a y\left(\frac{z^2}{2}\right)_0^a dy = \frac{a^2}{2}\int_0^a y\,dy$$

$$= \frac{a^2}{2}\left(\frac{y^2}{2}\right) = \frac{a^2}{2}\cdot\frac{a^2}{2} = \frac{a^4}{4} \longrightarrow (2)$$

$\longrightarrow$ SURFACE ABGF: $y=a$, $n=j$, $ds=dx\,dz$.

$$\iint_S \left((x^3-yz)i-2x^2yj+zk\right)\cdot n\,ds = \iint \left((x^3-az)i-2x^2aj+zk\right)\cdot j\,ds$$

$$= \int_0^a\int_0^a (-2x^2a)\,dx\,dz = -2a\int_0^a x^2(z)_0^a\,dx = -2a^2\int_0^a x^2\,dx$$

$$= -2a^2\left(\frac{x^3}{3}\right)_0^a = \frac{-2a^5}{3} \longrightarrow (3)$$

$\longrightarrow$ SURFACE OCDE: $y=0$, $n=-j$, $ds=dx\,dz$

$$\iint_S \left((x^3-yz)i-2x^2yj+zk\right)\cdot n\,ds = \iint (zk)\cdot(-j)\,dx\,dz = 0 \longrightarrow (4)$$

$\longrightarrow$ SURFACE OABC: $z=0$, $n=-k$, $ds=dx\,dy$

$$\iint_S \left((x^3-yz)i-2x^2yj+zk\right)\cdot n\,ds = \iint -(2x^2yj)\cdot-(k)\,dx\,dy = 0 \longrightarrow (5)$$

$\longrightarrow$ SURFACE DEFG: $z=a$, $n=k$, $ds=dx\,dy$

$$\iint_S \left((x^3-yz)i-2x^2yj+zk\right)\cdot n\,ds = \iint \left((x^3-ay)i-2x^2yj+ak\right)\cdot k\,ds$$

$$= \int_0^a\int_0^a a\,dx\,dy = a\int_0^a (y)_0^a\,dx = a^2\int_0^a dx = a^2[x]_0^a = a^3 \longrightarrow (7)$$

$$L.H.S = \iint_S \left((x^3-yz)i-2x^2yj+zk\right)\cdot n\,ds = a^5 - \frac{a^4}{4} + \frac{a^4}{4} - \frac{2a^5}{3} + a^3$$

$$= \frac{3a^5-2a^5}{3} + a^3 = \frac{a^5}{3} + a^3 \longrightarrow (B)$$

L.H.S $=$ R.H.S, $\iint_S F\cdot n\,ds = \iiint_V \nabla\cdot F\,dv$, $\therefore$ Gauss-Divergence Theorem is Verified.

05. Verify Divergence theorem for $\vec{F} = 4xz\,i - y^2 j + yz\,k$, taken over the cube bounded by $x=0, x=1, y=0, y=1$ and $z=0, z=1$.

Sol: By Gauss-Divergence Theorem,

$$\iint_S F\cdot n\,ds = \iiint_V \nabla\cdot F\,dv$$

$$\iint_S (4xz\,i - y^2 j + yz\,k)\cdot n\,ds = \iiint \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot F\,dv$$

$$R.H.S = \iiint_V \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(4xz\,i - y^2 j + yz\,k)\,dv$$

$$= \iiint \left[\frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2 j) + \frac{\partial}{\partial z}(yz)\right] dx\,dy\,dz$$

$$= \int_0^1\int_0^1\int_0^1 (4z - 2y + y)\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 (4z - y)\,dx\,dy\,dz$$

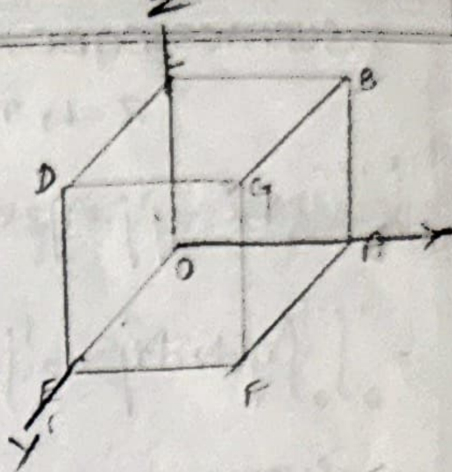
$$= \int_0^1\int_0^1 \left(\frac{4z^2}{2} - yz\right)_0^1 dx\,dy = \int_0^1\int_0^1 (2 - y)\,dx\,dy$$

$$= \int_0^1 \left(2y - \frac{y^2}{2}\right)_0^1 dx = \int_0^1 \left(2 - \frac{1}{2}\right) dx = \frac{3}{2}\int_0^1 1\,dx = \frac{3}{2}(x)_0^1$$

$$= \frac{3}{2}.$$

$$\therefore \iiint_V \nabla\cdot F\,dv = \frac{3}{2} \longrightarrow A$$

R.H.S: $\iint_S F.n\,ds$



→ SURFACE AFGB:

$x=1,\ n=i,\ ds=dydz$

$\int_0^1\int_0^1 (4xzi-y^2j+yzk)\cdot i\,dydz$

$=\int_0^1\int_0^1 (4zi-y^2j+yzk)\cdot i\,dydz = \int_0^1\int_0^1 4z\,dydz$

$=\int_0^1 \left(\frac{4z^2}{2}\right)_0^1 dy = \int_0^1 2\,dy = (2y)_0^1 = 2 \longrightarrow ①$

→ SURFACE OEDC:

$x=0,\ n=-i,\ ds=dydz$

$\int_0^1\int_0^1 (4xzi-y^2j+yzk)\cdot(-i)ds = \int_0^1\int_0^1 -(y^2j+yzk)\cdot-(i)dydz$

$=0. \longrightarrow ②$

→ SURFACE GDEF:

$y=1,\ n=j,\ ds=dx\,dz$

$\int_0^1\int_0^1 (4xzi-y^2j+yzk)\cdot j\,ds = \int_0^1\int_0^1 (4xzi-j+zk)\cdot j\,dxdz$

$= -\int_0^1 (z)_0^1 dx = -\int_0^1 dx = -1 \longrightarrow ③$

→ SURFACE OABC:

$y=0,\ n=-j,\ ds=dx\,dz$

$\int_0^1\int_0^1 (4xzi-y^2j+yzk)\cdot-(j)ds = \int_0^1\int_0^1 (4xzi)\cdot-j\,dxdz=0 \longrightarrow ④$

→ SURFACE OGBC:

$$z=1,\ n=k,\ ds=dx\,dy$$

$$\int_0^1\int_0^1 (4xzi - y^2j + yzk)\cdot k\,ds = \int_0^1\int_0^1 (4xi - y^2j + yk)\cdot k\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy = \int_0^1 \left(\frac{y^2}{2}\right)_0^1 dx = \int_0^1 \frac{1}{2}dx = \frac{1}{2}x = \frac{1}{2} \longrightarrow ⑤$$

→ SURFACE OAFE:

$$z=0,\ n=-k,\ ds=dx\,dy$$

$$\int_0^1\int_0^1 (4xzi - y^2j + yzk)\cdot -k\,dx\,dy = \int_0^1\int_0^1 (-y^2j)\cdot(-k)\,dx\,dy = 0 \longrightarrow ⑥$$

$$\therefore L.H.S = \iint_S F\cdot n\,ds = \frac{1}{2} + 0 + 2 + 0 - 1 + 0 = \frac{1}{2} + 1 = \frac{3}{2} \longrightarrow Ⓑ$$

From Ⓐ & Ⓑ,

$$L.H.S = R.H.S$$

$$\iint_S F\cdot n\,ds = \iiint_V \nabla\cdot F\,dv$$

$\therefore$ Gauss-Divergence Theorem is Verified.

Q6: Verify Divergence Theorem for $\vec{F} = (x^2-yz)i + (y^2-xz)j + (z^2-xy)k$ taken over a rectangular parallelopiped $0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c$.

Sol: Given, $\vec{F} = (x^2-yz)i + (y^2-xz)j + (z^2-xy)k$.

By Gauss Divergence Theorem,

$$\iint_S F\cdot n\,ds = \iiint_V \nabla\cdot F\,dv$$

$R.H.S = \iiint \nabla \cdot F \, dv,$

$$\nabla \cdot F = \left[i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right] \cdot \left[(x^2-yz)i + (y^2-zx)j + (z^2-xy)k \right]$$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x+2y+2z = 2(x+y+z).$$

$$\iiint_V \nabla \cdot F dv = \int_0^a \int_0^b \int_0^c 2(x+y+z)\,dx\,dy\,dz = 2\int_0^a \int_0^b \left(xz+yz+\frac{z^2}{2} \right)_0^c dx\,dy$$

$$= 2\int_0^a \int_0^b \left(xc+yc+\frac{c^2}{2} \right) dx\,dy = 2\int_0^a \left(xcy + \frac{cy^2}{2} + \frac{c^2}{2}y \right)_0^b dx$$

$$= 2\int_0^a \left(xcb + \frac{cb^2}{2} + \frac{c^2 b}{2} \right) dx = 2\left[\frac{x^2 cb}{2} + \frac{cb^2}{2}.x + \frac{c^2 b}{2}.x \right]_0^a$$

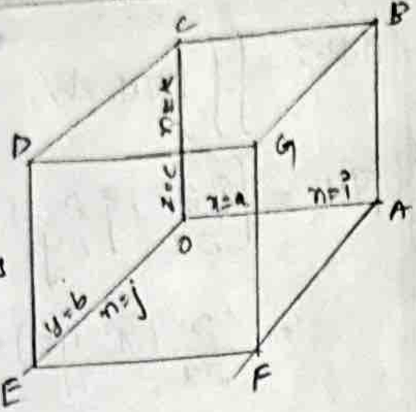
$$= 2\left[\frac{a^2 cb}{2} + \frac{cb^2 a}{2} + \frac{c^2 ba}{2} \right] = \frac{\cancel{2}abc}{\cancel{2}}(a+b+c)$$

$$\therefore \iiint_V \nabla \cdot F dV = abc(a+b+c) \longrightarrow (A)$$

$$R.H.S = \iint_S F.n\,ds = \iint_S \left((x^2-yz)i + (y^2-xz)j + (z^2-xy)k \right).n\,ds$$

Surface OGBC:

$z=c,\ n=k,\ ds=dx\,dy$



$$\iint\left((x^2-yz)i+(y^2-xz)j+(z^2-xy)k\right)\cdot n\,ds$$

$$=\int_{x=0}^{a}\int_{y=0}^{b}(x^2-yc)i+(y^2-xc)j+(c^2-xy)k\cdot k\,dx\,dy$$

$$=\int_{x=0}^{a}\int_{y=0}^{b}(c^2-xy)\,dx\,dy=\int_{x=0}^{a}\left((c^2y)_0^b-x\left(\frac{y^2}{2}\right)_0^b\right)dx$$

$$=\int_0^a\left(c^2b-\frac{xb^2}{2}\right)dx=\left(c^2bx\right)_0^a-\left(\frac{x^2}{2}\cdot\frac{b^2}{2}\right)_0^a=ac^2b-\frac{b^2a^2}{4}\longrightarrow ①$$

Surface OAFE: $z=0$ $n=-k,\ ds=dx\,dy$

$$\iint\left((x^2-yz)i+(y^2-xz)j+(z^2-xy)k\right)\cdot-(k)\,dx\,dy$$

$$=\int_0^a\int_0^b-(0-xy)\,dx\,dy=\int_0^a\int_0^b xy\,dx\,dy$$

$$=\int_0^a x\left(\frac{y^2}{2}\right)_0^b dx=\frac{b^2}{2}\int_0^a x\,dx=\left(\frac{x^2}{2}\right)_0^a\cdot\frac{b^2}{2}$$

$$=\frac{b^2a^2}{4}\longrightarrow ②$$

SURFACE AFGB:

$$x=a,\ n=i,\ ds=dydz$$

$$\iint((x^2-yz)i+(y^2-xz)j+(z^2-xy)k)\cdot n\,ds$$

$$=\iint((x^2-yz)i+(y^2-xz)j+(z^2-xy)k)\cdot i\,dy\,dz.$$

$$=\int_0^b\int_0^c(x^2-yz)\,dydz=\int_0^b\int_0^c(a^2-yz)\,dydz$$

$$=\int_0^b\left(a^2z-\frac{yz^2}{2}\right)_0^c dy=\int_0^b\left(a^2c-\frac{yc^2}{2}\right)dy=\left(a^2cy-\frac{y^2}{2}\cdot\frac{c^2}{2}\right)$$

$$=a^2cb-\frac{b^2c^2}{4}\longrightarrow③$$

SURFACE OEDC:

$$x=0,\ n=-i,\ ds=dy\,dz$$

$$\iint((x^2-yz)i+(y^2-xz)j+(z^2-xy)k)(-i)ds$$

$$=\int_0^b\int_0^c-(x^2-yz)\,dydz=\int_0^b\int_0^c yz\,dydz$$

$$=\int_0^b y\left(\frac{z^2}{2}\right)_0^c dy=\int_0^b\frac{yc^2}{2}dy=\left(\frac{y^2}{2}\right)_0^b\frac{c^2}{2}=\frac{b^2c^2}{4}\longrightarrow④$$

SURFACE GDEF: $y=b,\ n=j,\ ds=dxdz$

$$\iint((x^2-yz)i+(y^2-xz)j+(z^2-xy)k)\cdot j\,dx\,dz$$

$$\int_0^a\int_0^c (y^2-xz)\,dx\,dx = \int_0^a\int_0^c (b^2-xz)\,dx\,dz = \int_0^a \left(b^2z-\frac{xz^2}{2}\right)_0^c dx$$

$$\int_0^a \left(b^2c-\frac{xc^2}{2}\right)dx = \left(b^2cx-\frac{x^2}{2}\cdot\frac{c^2}{2}\right)_0^a = b^2ca-\frac{a^2c^2}{4} \longrightarrow (5)$$

SURFACE OABC:

$$y=0,\ n=-j,\ ds=dx\,dz$$

$$\iint \left((x^2-yz)i+(y^2-xz)j+(z^2-xy)k\right)\cdot(-j)\,ds$$

$$= \int_0^a\int_0^c -(y^2-xz)\,dx\,dz = \int_0^a\int_0^c xz\,dx\,dz = \int_0^a x\left(\frac{z^2}{2}\right)_0^c dx$$

$$= \left(\frac{x^2}{2}\frac{c^2}{2}\right)_0^a = \frac{c^2a^2}{4} \longrightarrow (6)$$

$$\therefore L.H.S = ac^2b-\frac{b^2a^2}{4}+\frac{a^2b^2}{4}+a^2bc-\frac{b^2c^2}{4}+\frac{b^2c^2}{4}+b^2ca-\frac{a^2c^2}{4}+\frac{a^2c^2}{4}$$

$$= ac^2b+a^2bc+b^2ca = abc(a+b+c). \longrightarrow (B)$$

From (A) & (B), $L.H.S = R.H.S$

$$\iint_S F\cdot n\,ds = \iiint_V \nabla\cdot F\,dV$$

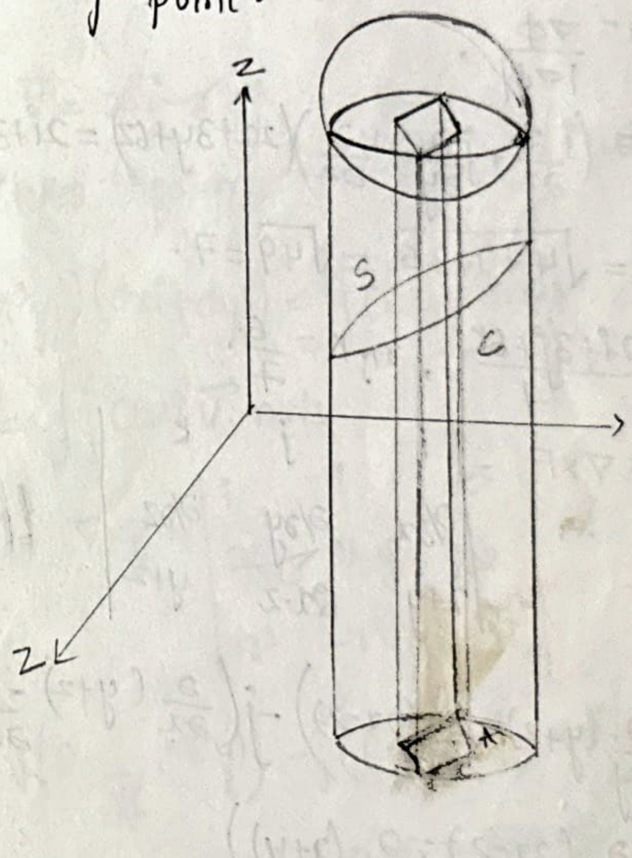
Hence Gauss - Divergence Theorem is verified.

STOKES THEOREM

STATEMENT: If "S" be an open surface bounded by closed curve "C", and $\vec{F}$ be any vector point function having continuous first order partial derivatives, then

$$\oint_C F.\vec{dr} = \iint_S \text{curl } F.\hat{n}\,ds = \iint_S \text{curl}F\,ds,$$

where 'n' is outward drawn unit normal to the surface at any point.



Q1: Apply Stokes Theorem to evaluate $\oint (x+y)dx + (y+z)dz + (2x-z)dy$ where 'c' is the boundary of triangle of vertices $(2,0,0), (0,3,0), (0,0,6)$.

Sol: By Stokes Theorem, $\oint_C \vec{F}\cdot\vec{dr} = \iint \text{curl}\, F \cdot n\, ds = \iint \nabla \times F$

n = equation of plane $\phi = 2x+3y+6z = 0$.

$\vec{F} = (x+y)i + (2x-z)j + (y+z)k$.

$n = \dfrac{\nabla\phi}{|\nabla\phi|}$,

$\nabla\phi = \left(i\dfrac{\partial}{\partial x} + j\dfrac{\partial}{\partial y} + k\dfrac{\partial}{\partial z}\right)(2x+3y+6z) = 2i+3j+6k$

$|\nabla\cdot\phi| = \sqrt{4+9+36} = \sqrt{49} = 7$.

$n = \dfrac{2i+3j+6k}{7}$, $n\cdot k = \dfrac{6}{7}$.

$$\text{Curl}\,F = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}(y+z) - \frac{\partial}{\partial z}(2x-z)\right) - j\left(\frac{\partial}{\partial x}(y+z) - \frac{\partial}{\partial z}(x+y)\right)$$

$$+ k\left(\frac{\partial}{\partial x}(2x-z) - \frac{\partial}{\partial y}(x+y)\right)$$

$= i(1+1) - j(0-0) + k(2-1) = 2i+k$.

$$\oint_C \vec{F}\cdot\vec{dr} = \iint \text{curl}\,F\cdot n\,ds = \iint (2i+k)\cdot\frac{2i+3j+6k}{7}\cdot\frac{dx\,dy}{|n\cdot k|}$$

$$\iint (2i+k)\cdot\frac{(2i+3j+6k)}{7}\,\frac{dx\,dy}{(6/7)} = \frac{1}{6}\iint (4+6)\,dx\,dy = \frac{10}{6}\int_0^2\int_0^3 dx\,dy$$

$$= \frac{10}{6}\int_0^2 (y)_0^3\,dx = \frac{10}{6}\times 3\int_0^2 dx = \frac{10}{6}\times 3\times 2 = \underline{10}$$

Q2: Verify Stokes Theorem for $\vec{V} = x^3 i - x^2 y j$, where 'c' is the boundary of rectangle whose sides are $x=0$, $x=3$, $y=0$, $y=4$ and the plane $z=0$.

Sol: Given,

$$\vec{V} = x^3 i - x^2 y j$$

By Stokes theorem, $\oint_C \vec{V}\cdot d\vec{r} = \iint_C \text{curl}\,V\cdot n\,ds$.

$$\oint_C (x^3 i - x^2 y j)(dx\,i + dy\,j) = \iint \text{curl}\,\vec{V}\cdot n\,ds$$

$$R.H.S = \iint \text{curl}\,\vec{V}\cdot n\,ds$$

$$\text{curl}\,\vec{V} = \nabla\times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^3 & -x^2 y & 0 \end{vmatrix}$$

$$= i\left(0 - \frac{\partial}{\partial z}(-x^2 y)\right) - j\left(0 - \frac{\partial}{\partial z}(x^3)\right) + k\left(\frac{\partial}{\partial x}(-x^2 y) - \frac{\partial}{\partial y}(x^3)\right)$$

$$= 0 - 0 + (-2xy - 0)k = \underline{-2xyk}.$$

'n' is unit normal to the surface $z=0$ is $n=k$.

$$\iint \text{curl}\,\vec{V}\cdot n\,ds = \iint \text{curl}\,\vec{V}\cdot n\,\frac{dx\,dy}{|n\cdot k|}$$

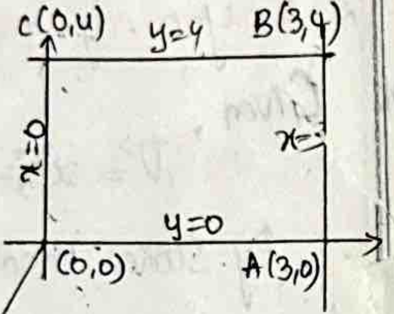
$$\iint -(2xy\hat{k})\cdot\hat{k}\,dx\,dy = \iint -2xy\,dx\,dy$$

$$-\int_0^3\int_0^4 2xy\,dx\,dy = -\int_0^3 2x\left(\frac{y^2}{2}\right)_0^4 dx = -\int_0^3 x(16-0)\,dx$$

$$= -\int_0^3 16x\,dx = -16\left(\frac{x^2}{2}\right)_0^3 = -16\frac{(9)}{2} = -72.$$

L.H.S : $\oint_C \vec{V}\cdot\vec{dr} = \int_{OA}\vec{V}\cdot\vec{dr} + \int_{AB}\vec{V}\cdot\vec{dr}$

$+ \int_{BC}\vec{V}\cdot\vec{dr} + \int_{CO}\vec{V}\cdot\vec{dr}.$



$$\int_C \vec{V}\cdot\vec{dr} = \int_C (x^3\hat{i} - x^2y\hat{j})(dx\hat{i} + dy\hat{j})$$

$$= \int_C x^3dx - x^2y\,dy$$

1* Along OA :- $y=0$ $x:(0\to 3)$
$dy = 0$

$$\int_{OA} x^3dx - x^2y\,dy = \int_0^3 (x^3)\,dx - 0 = \left(\frac{x^4}{4}\right)_0^3 = \frac{81}{4}.$$

2* Along AB :- $x=3$ $y:(0\to 4)$
$dx=0$

$$\int_{AB} x^3dx - x^2y\,dy = \int_0^4 -x^2y\,dy = -9\left[\frac{y^2}{2}\right]_0^4$$

$$= \frac{-9\times 16}{2} = -72.$$

3. Along BC: $y=4$ $\quad x:(3 \to 0)$

$dy=0$

$$\int_{BC} x^3dx - x^2y\,dy = \int_3^0 x^3dx = \left(\frac{x^4}{4}\right)_3^0 = -\frac{81}{4}.$$

4. Along CO: $x=0$ $\quad y:(4 \to 0)$

$dx=0$

$$\int_{CO} x^3dx - x^2y\,dy = \int_4^0 0 = 0.$$

$$L.H.S = \int x^3dx - x^2y\,dy = \frac{81}{4} - 72 - \frac{81}{4} + 0 = -72.$$

$\therefore$ L.H.S = R.H.S

Apply Green's theorem to Evaluate $\oint_c (y^2-2xy)\,dx + (x^2+y^2)\,dy$ where 'c' is the Closed Surface formed by $y=x^2$ and $x=y^2$

By Green's theorem we have,

$$\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial N}{\partial y}\right) dx\,dy$$

i.e $\oint_c (y^2-2xy)\,dx + (x^2+y)^2\,dy$

Here $M = y^2 = 2xy$, $N = x^2+y^2$

$$\frac{\partial M}{\partial y} = 2y-2x, \quad \frac{\partial N}{\partial x} = 2x$$

i.e $\iint_R (2x-2y+2x)\,dx\,dy$

$$= 2\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (2x-y)\,dy\,dx$$

$$= 2\int_{x=0}^{1} \left(2xy - \frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 2\int_0^1 \left[\left(2x^{3/2} - \frac{x}{2}\right) - \left(2x^3 - \frac{x^4}{2}\right)\right] dx$$

$$= 2\left[\frac{2x^{5/2}}{5/2} - \frac{x^2}{4} - \frac{2x^4}{4} + \frac{x^5}{10}\right]_0^1$$

$$= 2\left[\frac{4}{5} - \frac{1}{4} - \frac{1}{2} + \frac{1}{10}\right]$$

$$= \frac{3}{10}$$

(1) $\int_0^a \int_0^a xy\,dx\,dy$ · Ans $\frac{a^4}{4}$

(2) $\int_1^a \int_1^b \frac{dx\,dy}{xy}$

$\int_{y=1}^{y=b} [\log x]\,dy = \log b\,[\log y]_1^a$

$= \log b \log a$

$= \log ab$

(3) $\int_0^1 \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy$

$\int_{y=0}^{y=1} [\sin^{-1} x]_0^a\,dy = \sin^{-1} \pi/2\,[\sin^{-1} y] = \frac{\pi^2}{4}$

(4) $\int_1^2 \int_0^2 (x^3+y^3)\,dx\,dy = \frac{23}{2}$

(5) $\int_1^2 \int_0^3 (x^2y+xy^2)\,dx\,dy = 24$

(6) $\int_0^1 \int_0^{x^2} e^{y/x}\,dx\,dy$

$\int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$

$\int_0^1 xe^x = \int_0^1 [xe^x - x]_0^1 = -\frac{1}{2}+1 = \frac{1}{2}$

(7) $\int_0^1 \int_0^{1-x} e^{2x+3y}\,dx\,dy$

$\int_0^1 \left[\frac{e^{2x+3y}}{3}\right]_0^{1-x} dx = \frac{1}{3}\int_0^1 \left[e^{2x+3(1-x)} - e^{2x}\right] dy$

$= \frac{1}{3}\int_0^1 \left[e^{3-x} - e^{2x}\right] dx = \frac{1}{3}\left[\frac{e^{3-x}}{-1} - e^{2x}\right]_0^1 = \frac{1}{6}\left[2e^3 - 3e^2\right.$

(8) $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\, dy\, dx$

$\left[\frac{1}{\sqrt{a}} \tan^{-1}\frac{x}{a}\right]$

$\int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \cdot \tan^{-1}\frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}} dx$

$\int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \left[\tan^{-1} 1 - \tan^{-1} 0\right] dx$

$\frac{\pi}{4}\left[\sinh^{-1} x\right]_0^1 = \cancel{\frac{\pi}{4}}\pi/2 = \frac{\pi}{4}\left[\sinh^{-1}(1)\right]$

(9) $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx$ $\quad \int \sqrt{a^2-x^2}\,dx = \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$

$= \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{(a^2-x^2)}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$

$= \int_0^a \left[\underbrace{\frac{\sqrt{a^2-x^2}}{2}\sqrt{a^2-x^2-\cancel{y^2}}}_{(0)} + \frac{(a^2-x^2)}{2}\sin^{-1}\frac{\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}}\right] dx$

$= \int_0^a \frac{a^2-x^2}{2}\, \pi/2\, dx$

$= \frac{\pi}{4}\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{\pi a^3}{6}$

(10) $\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dx\, dy\, dz$

$\int_0^1 \int_0^1 \int_0^1 e^x \cdot e^{y+z} = (e-1)^3$

$= (e^1-1)^3$

(11) $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz = \frac{1}{48}$

(12) $\int_1^3 \int_{1/x}^{1} \int_0^{\sqrt{xy}} xyz \, dx\, dy\, dz$ (2)

$\frac{1}{2}\int_1^3 \int_{1/x}^{1} x^2y^2 \, dy\, dz$

$\frac{1}{6}\int_1^3 x^2 \left(\cancel{\tfrac{1}{x^3}} \; (1 - 1/x^3) \right)$

$\frac{1}{6}\int_1^3 \left[x^2 - \frac{1}{x}\right] dx = \frac{1}{6}\left[\frac{x^3}{3} - \log x\right]_1^3 = 1/6\left[\frac{27}{3} - \log 3 - \frac{1}{3} + \log 1\right]$

$= \frac{1}{6}\left[\frac{26}{3} - \log 3\right]$

(13) $\int_{-1}^{1} \int_0^{z} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz = 0$

(14) $\int_1^e \int_1^{\log y} \int_1^{e^x} \log z \, dz\, dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{x=\log y} \int_{z=1}^{z=e^x} \cancel{z \log z - 1]} \log z \, dx\, dy\, dz$

$\int_{y=1}^{e} \int_{x=1}^{x=\log y} \left[z \log z - z\right]_1^{e^x} dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{x=\log y} \left[e^x \log e^x - e^x\right]_1^{e^x} dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{x=\log y} \cancel{e^x\left[\log e^x - 1\right] dx} \left[e^x\left[\log e^x - 1\right] - 1\left[\log 1 - 1\right]\right] dx\, dy$

$\int_{y=1}^{e} \int_{x=1}^{\log y} \left[e^x\left[x-1\right]+1\right] dx\, dy$

$\int_1^e \int_1^{\log y} \left[xe^x - e^x + 1\right) dx\, dy = \int \left[xe^x - e^x \cancel{+} e^x + 1\right]_1^{\log y} dy$

$\cancel{\int_1^e \left[\log y \, e^{y} \ldots\right]} dy$