UNIT-II
Calculus One Variable

Rolles theorem

Statement:— Let a function $f(x)$ be defined in $[a,b]$ such that

① $f(x)$ is continuous in $[a,b]$
② $f(x)$ is derivable in (a,b)
③ $f(a)=f(b)$ then $\exists\, c \in (a,b)$ such that

$$f'(c)=0.$$

Lagranges Mean value theorem

Statement:— Let a function $f(x)$ be defined in $[a,b]$ such that

① $f(x)$ is continuous in $[a,b]$
② $f(x)$ is derivable in (a,b) then $\exists\, c \in (a,b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(c)$

Cauchys Mean value theorem:—

Statement:— Let $f(x)$ & $g(x)$ be two functions be defined in $[a,b]$ such that

① $f(x)$ & $g(x)$ are continuous in $[a,b]$
② $f(x)$ & $g(x)$ are derivable in (a,b)
③ $g'(x) \neq 0 \ \forall x \in [a,b]$ then there exist $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Applications :-

1. Verify Rolles theorem for the following function $f(x) = x^2(1-x^2)^2$ in $[0,1]$

Sol :- $f(x) = x^2(1-x)^2$ in $[0,1]$

$f(x)$ is continuous in $[0,1]$

$f'(x) = x^2 \, 2(1-x)(-1) + 2x(1-x^2)^2$

$f(x)$ is derivable in $[0,1)$

$f(0) = 0^2 \cdot (1-0)^2 = 0$

$f(1) = 1^2(1-1)^2 = 0$

$\therefore f(0) = f(1)$

All the conditions of Rolles theorem satisfied. Then there exist $c \in (a,b)$ such that $f'(c) = 0$.

$\Rightarrow c^2 \, 2(1-c)(-1) + 2c(1-c^2)^2 = 0$

$\Rightarrow 2c(1-c)[-c+1-c] = 0$

$\Rightarrow -2c+1 = 0$, $c = 0$ & c , $c=0$, $1-c=0$, $c=0$ $c=1$

$\Rightarrow c = 1/2 = 0.5 \in (0,1)$

$\therefore c \in (0,1)$

Rolle's theorem is satisfied

∴ Verify Rolles theorem for the function
$f(x) = (x-a)^m (x-b)^n$ in $[a,b]$

Sol $f(x) = (x-a)^m (x-b)^n$ in $[a,b]$

$f(x)$ is continuous in $[a,b]$

$f'(x) = m(x-a)^{m-1}(x-b)^n + (x-a)^m n(x-b)^{n-1}$

$f(x)$ is derivable in (a,b)

$f(a) = (a-a)^m (a-b)^n = 0$

$f(b) = (b-a)^m (b-b)^n = 0$

$\therefore f(a) = f(b)$

∴ All the conditions of Rolles theorem Satisfied. then there exist $c \in (a,b)$ such that $f'(c) = 0$

i.e., $m(c-a)^{m-1}(c-b)^n + (c-a)^m n(c-b)^{n-1} = 0$

$\Rightarrow (c-a)^{m-1}(c-b)^{n-1}[m(c-b) + n(c-a)] = 0$

$\Rightarrow mc - mb + nc - na = 0$

$mc + nc = mb + na.$

$\Rightarrow c(m+n) = mb + na$

$\therefore c = \dfrac{mb+na}{m+n}.$

$c \in (a,b)$

∴ Rolle's theorem is Verified.

3. Verify Rolles theorem for the function
$f(x) = x^3$ in $[1,3]$

Sol:- $f(x) = x^3$ in $[1,3]$

$f(x)$ is continuous in $[1,3]$

$f'(x) = 3x^2$

$f(x)$ is derivable in $(1,3)$

$f(1) = 1^3 = 1$

$f(3) = 3^3 = 27$

$f(1) \neq f(3)$.

Rolle's theorem is not applicable for the given function.

4. Verify Rolles theorem for the function
$f(x) = \sin x \sqrt{\cos 2x}$ in $[0, \pi/4]$

Sol $f(x) = \sin x \sqrt{\cos 2x}$ in $[0, \pi/4]$

$f(x)$ is continuous in $[0, \pi/4]$

$f'(x) = \sin x \cdot \frac{1}{\cancel{2}\sqrt{\cos 2x}} (-\cancel{2}\sin 2x) + \sqrt{\cos 2x} \cos x$

$f(x)$ is derivable in $(0, \pi/4)$

$f(0) = \sin 0 \sqrt{\cos 2(0)} = 0$

$f(\pi/4) = \sin \pi/4 \sqrt{\cos \cancel{2} \cdot \frac{\pi}{\cancel{4}_2}} = 0$.

$f(0) = f(\pi/4)$

$\therefore$ All the conditions of Rolles theorem are satisfied then there exist $c \in (0, \pi/4)$ such that $f'(c) = 0$

i.e., $\frac{-\sin x \sin 2x}{\sqrt{\cos 2c}} + \cos c\sqrt{\cos 2c} = 0$

$\Rightarrow \frac{-\sin c \sin 2c}{\sqrt{\cos 2c}} = -\cos c\sqrt{\cos 2c}$

$\Rightarrow \cancel{-}\sin c \sin 2c = \cancel{-}\cos c \cos 2c$

$\Rightarrow \cos c \cos 2c - \sin c \sin 2c = 0$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$.

$\Rightarrow \cos(c+2c) = 0$

$\Rightarrow \cos 3c = 0$

$\Rightarrow 3c = \cos^{-1} 0$

$\Rightarrow 3c = \pi/2$

$\Rightarrow c = \pi/6$.

$\therefore c = \pi/6$.

$\therefore c \in (0, \pi/4)$

Rolles theory is verified.

5 Verify Lagranges mean value theorem for the function $f(x) = x^3 - x^2 - 5x + 3$ in $[0, 4]$

Sol $f(x) = x^3 - x^2 - 5x + 3$ in $[0, 4]$

$f(x)$ is continuous in $[0, 4]$

$f'(x) = 3x^2 - 2x - 5$

$f(x)$ is derivable in $(0, 4)$

$\therefore$ All the conditions of L.M.V theorem are satisfied then there exist $c \in (0, 4)$ such that

$$\frac{f(4) - f(0)}{4-0} = f'(c)$$

$f(4) = 4^3 - 4^2 - 20 + 3$
$= 64 - 16 - 20 + 3$
$= 67 - 36$
$= \underline{\underline{31}}$

$$\rightarrow \frac{31-3}{4} = 3c^2 - 2c - 5$$

$$\Rightarrow \frac{28}{4} = 3c^2 - 2c - 5$$

$$\Rightarrow 7 = 3c^2 - 2c - 5$$

$$\Rightarrow \boxed{3c^2 - 2c - 12 = 0}$$

$$\Rightarrow c = \frac{2 \pm \sqrt{4 - 4(3)(-12)}}{6}$$

$$\Rightarrow c = \frac{2 \pm \sqrt{4+144}}{6}$$

$$\Rightarrow c = \frac{2 \pm \sqrt{148}}{6}$$

$$\Rightarrow c = \frac{2+\sqrt{148}}{6}, \frac{2-\sqrt{148}}{6}$$

12.165
+ 2
6) 14.165 (2.36
12
21
18
36
36
50

$$\Rightarrow c = 2.360$$

$$\Rightarrow c \in (0, 4)$$

Lagranges mean value theorem is Satisfied

13/9/2011

1. Verify L.M.V theorem $f(x) = x^3$ in $[a, b]$

Sol $f(x) = x^3$ in $[a, b]$

1. $f(x)$ is continuous in $[a, b]$

2. $f'(x) = 3x^2$

$\therefore f(x)$ is derivable in (a, b)

$\therefore$ All the conditions of LMVT are satisfied, then there exist $c \in (a, b)$ such that

$$\Rightarrow \frac{f(b) - f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{b^3 - a^3}{b-a} = 3c^2$$

$$\Rightarrow \frac{\cancel{(b-a)}(b^2 + ab + a^2)}{\cancel{b-a}} = 3c^2$$

$$\Rightarrow b^2 + ab + a^2 = 3c^2$$

$$\Rightarrow c^2 = \frac{a^2 + ab + b^2}{3}$$

$$\Rightarrow c = \sqrt{\frac{a^2 + ab + b^2}{3}}.$$

$\therefore c \in (a, b)$

2. Using L.M.V.T. Show that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Sol Let $f(x) = \tan^{-1}(x)$ & Let $[a, b]$ be the interval

① $f(x)$ is continuous in $[a, b]$

② $f(x)$ is derivable in (a, b)

$\therefore$ All the conditions of LMVT are satisfied then there exist $c \in (a,b)$ such that

$$f'(x) = \frac{1}{1+x^2}$$

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{\tan^{-1}(b) - \tan^{-1}(a)}{b-a} = \frac{1}{1+c^2}$$

$c \in (a, b)$

$\Rightarrow$ i.e., $a < c < b$.

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1+a^2 < 1+c^2 < 1+b^2$$

$$\Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\Rightarrow \frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

$$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

$$\Rightarrow \tan^{-1}a + \frac{b-a}{1+b^2} < \tan^{-1}b < \tan^{-1}a + \frac{b-a}{1+a^2}$$

Compare with the given eqns.

$$\Rightarrow \tan^{-1}a = \pi/4 \qquad \tan^{-1}b = \tan^{-1}4/3$$

$$a = 1 \qquad b = 4/3$$

$$\Rightarrow \frac{\pi}{4} + \frac{\frac{4}{3}-1}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} < \pi/4 + \frac{\frac{4}{3}-1}{1+1}$$

$$\Rightarrow \frac{\pi}{4} + \frac{\frac{1}{3}}{\frac{25}{9}} < \tan^{-1}4/3 < \pi/4 + \frac{\frac{1}{3}}{2}$$

$$\Rightarrow \pi/4 + \frac{3}{25} < \tan^{-1}(4/3) < \pi/4 + \frac{1}{6}$$

Using LMVT prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}$

Sol Let $f(x) = \sin^{-1}x$ & let $[a,b]$ be the Interval

$f(x)$ is continuous in $[a,b]$

$f(x)$ is derivable in (a,b)

$\therefore$ All the conditions of LMVT are satisfied then there exist $c \in (a,b)$ such that

$$\Rightarrow \frac{f(b)-f(a)}{b-a} = f'(c)$$

$$\Rightarrow \frac{\sin^{-1}b - \sin^{-1}a}{b-a} = \frac{1}{\sqrt{1-c^2}}$$

$c \in (a,b)$

$\Rightarrow$ i.e., $a < c < b$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow -a^2 > -c^2 > -b^2$$

$$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$$

$$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$$

$$\Rightarrow \sin^{-1}a + \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b < \sin^{-1}a + \frac{b-a}{\sqrt{1-b^2}}$$

compare with the given eqn.

$\sin^{-1} a = \pi/6$ $\qquad$ $\sin^{-1} b = \sin^{-1} 3/5$

$b = \frac{3}{5}$

$a = \frac{1}{2}$.

$\therefore \frac{\pi}{6} + \frac{\frac{3}{5} - \frac{1}{2}}{\sqrt{1-\frac{1}{4}}} < \sin^{-1} 3/5 < \sin^{-1}\left(\frac{1}{2}\right) + \frac{\frac{3}{5} - \frac{1}{2}}{\sqrt{1-\frac{9}{25}}}$

$\therefore \frac{\pi}{6} + \frac{\frac{6-5}{\cancel{10}\,5}}{\frac{\sqrt{3}}{\cancel{2}}} < \sin^{-1} 3/5 < \pi/6 + \frac{\frac{6-5}{\cancel{10}\,2}}{\frac{\sqrt{16}}{\cancel{5}}}$

$\therefore \frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1} 3/5 < \pi/6 + \frac{1}{8}$.

4. Using LMVT prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} \ngtr \cos^{-1} 3/5 > \frac{\pi}{3} - \frac{1}{8}$

Sol Let $f(x) = \cos^{-1}x$ & Let $[a,b]$ be the Interval

$f(x)$ is continuous in $[a,b]$

$f(x)$ is derivable in (a,b)

$\therefore$ All the conditions of LMVT is satisfied then

there exist $c \in (a,b)$ such that

$\Rightarrow \frac{f(b) - f(a)}{b-a} = f'(c)$

$\Rightarrow \frac{\cos^{-1}(b) - \cos^{-1}(a)}{b-a} = \frac{-1}{\sqrt{1-c^2}}$

$c \in (a,b)$

ie., $a < c < b$

$\Rightarrow a^2 \ngtr c^2 \nless b^2$

$\Rightarrow 1-a^2 \ngtr 1-c^2 \ngtr 1-b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1} b - \cos^{-1} a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-(b-a)}{\sqrt{1-a^2}} > \cos^{-1} b - \cos^{-1} a > \frac{-(b-a)}{\sqrt{1-b^2}}$$

$$\Rightarrow \cos^{-1} a - \frac{(b-a)}{\sqrt{1-a^2}} > \cos^{-1} b > \cos^{-1} a - \frac{(b-a)}{\sqrt{1-b^2}}$$

compare with the given eqn.

$\cos^{-1} a = \frac{\pi}{3}$. $\qquad \cos^{-1} b = \cos^{-1} 3/5$

$\qquad b = 3/5$

$a = \frac{1}{2}$

$$\therefore \frac{\pi}{3} - \frac{\left(\frac{3}{5} - \frac{1}{2}\right)}{\sqrt{1-\frac{1}{4}}} > \cos^{-1} 3/5 > \frac{\pi}{3} - \frac{\left(\frac{3}{5} - \frac{1}{2}\right)}{\sqrt{1-\frac{9}{25}}}.$$

$$\frac{\pi}{3} - \frac{\frac{(6-5)}{10\cancel{5}}}{\frac{\sqrt{3}}{\cancel{2}}} > \cos^{-1} 3/5 > \frac{\pi}{3} - \frac{\frac{(6-5)}{10}}{\frac{\sqrt{16}}{\cancel{5}}}$$

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1} 3/5 > \frac{\pi}{3} - \frac{1}{8}$$

16/7/11

1. If $f(x) = \frac{1}{x^2}$ & $g(x) = \frac{1}{x}$ in $[a, b]$ then show that the value of 'c' of the cauchy's mean value theorem is harmonic mean.

Sol:- $f(x) = \frac{1}{x^2}$ & $g(x) = \frac{1}{x}$ in $[a, b]$

① $f(x)$ & $g(x)$ are continuous in $[a, b]$

② $f'(x) = \frac{-2}{x^3}$, $g'(x) = \frac{-1}{x^2}$

$f(x)$ & $g(x)$ are derivable in (a, b)

③ $g'(x) \neq 0 \ \forall x \in (a, b)$.

∴ All the conditions of C.M.V.T satisfied. then there exist $c \in (a, b)$ such that

$$\Rightarrow \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\frac{1}{b^2} - \frac{1}{a^2}}{\frac{1}{b} - \frac{1}{a}} = \frac{\frac{-2}{c^3}}{\frac{-1}{c^2}}$$

$$\Rightarrow \frac{\frac{a^2 - b^2}{a^2 b^2}}{\frac{a-b}{ab}} = \frac{2}{c}$$

$$\Rightarrow \frac{(a+b)\cancel{(a-b)}}{(ab)^{\cancel{2}}} \times \frac{\cancel{ab}}{\cancel{a-b}} = \frac{2}{c}$$

$$\Rightarrow \frac{a+b}{ab} = \frac{2}{c}$$

$$\Rightarrow c = \frac{2ab}{a+b}$$

∴ c is a harmonic mean

2. Find the value of 'c' of Cauchy's mean value theorem for the functions $f(x) = e^x$ & $g(x) = e^{-x}$ in $[a,b]$.

Sol: Given $f(x) = e^x$ & $g(x) = e^{-x}$ in $[a,b]$

$f(x)$ & $g(x)$ are continuous in $[a,b]$

$f'(x) = e^x$ & $g'(x) = -e^{-x}$ in $[a,b]$

$f(x)$ & $g(x)$ are derivable in (a,b).

$g'(x) \neq 0 \; \forall x \in (a,b)$.

$\therefore$ All the conditions of CMVT is satisfied. then there exist $c \in (a,b)$. Such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

$$\Rightarrow \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{e^c}{-e^{-c}}$$

$$\Rightarrow \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} = \frac{e^c}{-\frac{1}{e^c}}$$

$$\Rightarrow \frac{e^b - e^a}{\frac{e^a - e^b}{e^a \cdot e^b}} = -\frac{e^c}{1} \times \frac{e^c}{1}$$

$$\Rightarrow \frac{e^b - e^a}{1} \times \frac{e^a \cdot e^b}{e^a - e^b} = -e^{2c}$$

$$\Rightarrow -(e^a - \cancel{e^b}) \times \frac{e^a \cdot e^b}{e^a - \cancel{e^b}} = -e^{2c}$$

$\Rightarrow -e^a \cdot e^b = -e^{2c}$

$\Rightarrow -e^{a+b} = -e^{2c}$.

Since the bases are same powers can be equated.

$\Rightarrow 2c = a+b$

$\Rightarrow c = \frac{a+b}{2} \in (a,b)$

3. Verify Cauchy's mean value theorem for the fn. $f(x) = x^2$ & $g(x) = x^3$ in $[1,2]$

<u>Sol</u>: $f(x) = x^2$ & $g(x) = x^3$ in $[1,2]$

(i) $f(x)$ & $g(x)$ are continuous in $[1,2]$

(ii) $f'(x) = 2x$ & $g'(x) = 3x^2$

$f(x)$ & $g(x)$ are derivable in $(1,2)$

(iii) $g'(x) \neq 0 \ \forall x \in (1,2)$

$\therefore$ All the conditions of CMVT is satisfied.

then there exist $c \in (1,2)$ such that

$$\Rightarrow \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

$$\Rightarrow \frac{b^2-a^2}{b^3-a^3} = \frac{2\cancel{c}}{3c^{\cancel{2}}}$$

$$\Rightarrow \frac{4-1}{8-1} = \frac{2}{3c}$$

$$\Rightarrow \frac{3}{7} = \frac{2}{3c}$$

$\Rightarrow 9c = 14 \Rightarrow c = \frac{14}{9} = 1.55 \simeq 1.6$.

$\therefore c \in (1,2)$ CMVT is verified.

9) 14 (1.55
9
50
45
5

4. find c of Cauchy's mean value theorem for the following fns. $f(x) = \sin x$ & $g(x) = \cos x$ in $[a, b]$

Sol $f(x) = \sin x$ & $g(x) = \cos x$ in $[a,b]$

① $f(x)$ & $g(x)$ are continuous in $[a,b]$

② $f'(x) = \cos x$ & $g'(x) = -\sin x$.

$f(x)$ & $g(x)$ are derivable in (a,b)

③ $g'(x) \neq 0 \ \forall x \in (a,b)$

∴ All the conditions of CMVT satisfied. then there exist $c \in (a,b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{\sin b - \sin a}{\cos b - \cos a} = \frac{\cos c}{-\sin c}$$

$$\Rightarrow \frac{\cancel{2} \cos \frac{b+a}{2} \cancel{\sin \frac{b-a}{2}}}{+ \cancel{2} \sin \frac{b+a}{2} \cancel{\sin \frac{b-a}{2}}} = + \frac{\cos c}{\sin c}$$

$$\Rightarrow \cot \frac{b+a}{2} = \cot c$$

$$c = \frac{a+b}{2}$$

16/09/2011

3. Taylor's Series, Expansion of function in power Series:-

Taylor's Series:-

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \frac{h^4}{4!} f^{iv}(a)$$
$$+ \cdots\cdots + \quad \text{[If boundary value given]}$$

Maclarins Series:-

$$f(x) = f(0) + h f'(0) + \frac{h^2}{2!} f''(0) + \frac{h^3}{3!} f'''(0) + \frac{h^4}{4!} f^{iv}(a)$$
$$+ \cdots\cdots \quad \text{[If boundary value not given]}$$

1. Find Taylor series expansion of $\sin 2x$ about $x = \pi/4$. (Or) obtain the power series of $\sin 2x$ in power of $(x - \pi/4)$

Sol:- $f(x) = \sin 2x$.

Taylor Series:-

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

$f(\ldots$ [illegible] $-\pi/4) f'(\pi/4) + \frac{(x-\pi/4)^2}{2!}$ [illegible]

[illegible] $(\pi/4) + \frac{(x-\pi/4)^3}{3!} f'''(\pi/4) + \cdots \rightarrow (1)$

[illegible] $= \sin \pi/2 = 1$

[illegible] $(\pi/4) = 2\cos \pi/2 = 0.$

[illegible] $(\pi/4) = -4 \sin \pi/2 = -4.$

$f''(\pi/4) =$ [illegible] $\cos \pi/2 = 0$

Now Subst. in (I).

$$\sin 2x = 1 - \frac{4(x-\pi/4)^2}{2!} + \frac{16(x-\pi/4)^4}{4!} + \cdots$$

22/7/2011

2. Taylor's Series to expand $2x^3+7x^2-x-6$ in powers of $(x-2)$

Sol Let $f(x) = 2x^3+7x^2-x-6$

By taylor's Series.

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

$$f(x) = f[\underset{a}{2} + \overset{h}{x-2}] = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) \rightarrow (1)$$

$f(x) = 2x^3+7x^2-x-6.$

$f(2) = 2(2)^3+7(2)^2-2-6 = 16+28-2-6 = 36$

$f'(x) = 6x^2+14x-1$

$f'(2) = 6(2)^2+14(2)-1 = 24+28-1 = 51.$

$f''(x) = 12x+14. \Rightarrow f''(2) = 24+14 = 38$

$f'''(x) = 12 \Rightarrow f'''(2) = 12$

$$2x^3+7x^2-x-6 = 36+51(x-2)+38\frac{(x-2)^2}{2!}+12\frac{(x-2)^3}{3!}$$

3. Expand $\sin x$ in ~~expandi~~ Ascending power of $(x-\pi/2)$

Sol Let $f(x) = \sin x$:

By Taylor's Series

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots$$

~~$f(\sin x)$~~ $f(x) = f(\underset{a}{\pi/2} + \underset{h}{x-\pi/2})$

$= f(\pi/2) + (x-\pi/2) f'(\pi/2) + \frac{(x-\pi/2)^2}{2!} f''(\pi/2) + \frac{(x-\pi/2)^3}{3!} f'''(\pi/2) + \cdots \rightarrow (1)$

$f(x) = \sin x \Rightarrow f(\pi/2) = \sin \pi/2 = 1$

$f'(x) = \cos x \Rightarrow f'(\pi/2) = \cos \pi/2 = 0$

$f''(x) = -\sin x \Rightarrow f''(\pi/2) = -\sin \pi/2 = -1$

$f'''(x) = -\cos x \Rightarrow f'''(\pi/2) = -\cos \pi/2 = 0.$

$f^{iv}(x) = \sin x \Rightarrow f^{iv}(\pi/2) = \sin \pi/2 = 1$ Subst in (1)

$\sin x = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \cdots\cdots$

4. Show that $\log \sin(x+h) = \log \sin x + h \cot x - \frac{h^2}{2!} \operatorname{cosec}^2 x + \frac{h^3}{3!} 2 \operatorname{cosec}^2 x \cot x + \cdots\cdots$

Sol Let $f(x+h) = \log \sin(x+h)$

$f(x) = \log \sin x.$

By Taylor's Series.

$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) \cdots\cdots$

$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$

$f(x) = \log \sin x.$

$f'(x) = \frac{1}{\sin x} \cos x = \cot x.$

$f''(x) = -\operatorname{cosec}^2 x$

$f'''(x) = +2 \operatorname{cosec}^2 x \cot x.$

$\log \sin(x+h) = \log \sin x + h \cot x - \frac{h^2}{2!} \operatorname{cosec}^2 x + \frac{h^3}{3!} 2 \operatorname{cosec}^2 x \cot x + \cdots\cdots$

5. Expand $\sin x$ in a infinite Series.

Sol Let $f(x) = \sin x$.

By Maclarins Series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \cdots \rightarrow (1)$$

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f^{IV}(x) = \sin x \Rightarrow f^{IV}(0) = \sin 0 = 0$

$f^{V}(x) = \cos x \Rightarrow f^{V}(0) = \cos 0 = 1$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$$

6. Expand $\log(\sec x)$. in a infinite Series.

Sol Let $f(x) = \log(\sec x)$

By Maclarins Series.

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots\cdots \rightarrow (1)$$

$f(x) = \log(\sec x) \Rightarrow f(0) = 0$

$f'(x) = \frac{\sec x \tan x}{\sec x} \Rightarrow f'(0) = 0$

$f''(x) = \sec^2 x \Rightarrow f''(0) = 1$

$f'''(x) = 2\sec^2 x \tan x \Rightarrow f'''(0) = 0$

$f^{IV}(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \Rightarrow f^{IV}(0) = 0 + 2 = 2$

$f^{V}(x) = 8\sec^4 x \cdot \tan x \Rightarrow f^{V}(0) = 0$

$f^{VI}(x) = 32\sec^4 x \tan^2 x + 8\sec^6 x \Rightarrow f^{VI}(0) = 8$

$\log(\sec x) = \frac{x^2}{2!} + \frac{2x^4}{4!} + \frac{8x^6}{6!} + \cdots$

7. write Expansion of $\cos^2 x$.

Sol Let $f(x) = \cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$.

By Maclarins series.

$f(x) = f(0) + x f'(x) + \frac{x^2}{2!} f''(x) + \frac{x^3}{3!} f'''(x) + \cdots \to (1)$

$f(x) = \frac{1}{2} + \frac{\cos 2x}{2} \Rightarrow f(0) = 1$

$f'(x) = -\sin 2x \Rightarrow f'(0) = 0$

$f''(x) = -2\cos 2x \Rightarrow f''(0) = -2$

$f'''(x) = 4\sin 2x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = 8\cos 2x \Rightarrow f^{iv}(0) = 8$

$\therefore \cos^2 x = 1 - 2\cdot\frac{x^2}{2!} + 8\frac{x^4}{4!} + \cdots$

1. S.T $\log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \cdots$

2. Exp $\cos x$.

3. Exp. $\sin^2 x$.

1. S.T $\log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \cdots$

Sol Let $f(x) = \log(1+x)$.

By Maclarins series.

$f(x) = f(0) + x f'(x) + \frac{x^2}{2!} f''(x) + \frac{x^3}{3!} f'''(x) + \cdots \to (1)$

$f(x) = \log(1+x) \Rightarrow f(0) = 0$

$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$

$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$

$f'''(x) = +2/(1+x)^3 \Rightarrow f'''(0) = +2$

$\log(1+x) = x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \cdots$

2. Expand $\cos x$

Sol Let $f(x) = \cos x$.

By Maclarins Series.

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} + f'''(0) \ldots\ldots$

$f(x) = \cos x \Rightarrow f(0) = 1$

$f'(x) = -\sin x \Rightarrow f'(0) = 0$

$f''(x) = -\cos x \Rightarrow f''(0) = -1$

$f'''(x) = \sin x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\ldots$

3. Expand. $\sin^2 x$.

Sol Let $f(x) = \sin^2 x = \frac{1}{2} - \frac{\cos 2x}{2}$

By Maclarins Series.

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots\ldots$

$f(x) = \frac{1}{2} - \frac{\cos 2x}{2} \Rightarrow f(0) = 0.$

$f'(x) = \sin 2x \Rightarrow f'(0) = 0$

$f''(x) = 2\cos 2x \Rightarrow f''(0) = 2$

$f'''(x) = -4\sin 2x \Rightarrow f'''(0) = 0$

$f^{iv}(x) = -8\cos 2x \Rightarrow f^{iv}(0) = -8$

$f^{v}(x) = +16\sin 2x \Rightarrow f^{v}(0) = 0$

$f^{vi}(x) = 32\cos 2x \Rightarrow f^{vi}(0) = 32$

$\therefore \sin^2 x = 2\frac{x^2}{2!} - 8\frac{x^4}{4!} + 32\frac{x^6}{6!} - \ldots\ldots$

22/9/2011 4. Curvature & Radius of Curvature.

Curvature: The curvature is rate of change of bendness of the curve.

→ The curvature of a straight line at every point is '0'.

→ The curvature of a circle is constant and is equal to the reciprocal of the radius.

Radius of Curvature: If the curvature of a curve at any point is non-zero. then its reciprocal is defined as the radius of the curvature of the curve at that point.

→ It is denoted by "ρ".

23/9/2011

I Radius of Curvature in Cartesian form:

→ Let the equation of curve be $y = f(x)$ then Radius of curvature $\rho = \dfrac{[1+y_1^2]^{3/2}}{y_2} = \dfrac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$

II Radius of Curvature in polar form:

→ Let the Equation of the curve $x = f(\theta)$ & $y = g(\theta)$. Then Radius of curvature

$$\rho = \frac{\left[[f'(\theta)]^2 + [g'(\theta)]^2\right]^{3/2}}{f'(\theta)g''(\theta) - g'(\theta)f''(\theta)}$$

III Radius of curvature in polar coordinates:

→ Let the equation of the curve be $r = f(\theta)$ then Radius of curvature $\rho = \dfrac{[r^2+r_1^2]^{3/2}}{r^2+2r_1^2-rr_2}$ where $r_1 = f'(\theta)$, $r_2 = f''(\theta)$

1. Find the radius of Curvature $xy = c$ at (x, y).

Sol: Radius of Curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2} \rightarrow (1)$

$\Rightarrow y = c/x$.

D. w.r.t o x.

$\Rightarrow y_1 = \frac{-c}{x^2}$

$\Rightarrow y_2 = \frac{2c}{x^3}$. Subst. in (1)

$\Rightarrow \rho = \frac{\left[1+\frac{c^2}{x^4}\right]^{3/2}}{\frac{2c}{x^3}}$

$\Rightarrow \rho = \frac{(x^4+c^2)^{3/2}}{x^6} \times \frac{x^3}{2c}$

$\Rightarrow \rho = \frac{[x^4+c^2]^{3/2}}{2cx^3}$.

2. Find the radius of Curvature $y^2 = 4ax$.

Sol:- Radius of Curvature. $\rho = \frac{[1+y_1^2]^{3/2}}{y_2} \rightarrow (1)$

$\Rightarrow y^2 = 4ax$.

D. w.r.t o x

$y = 2\sqrt{ax}$

$\Rightarrow 2yy_1 = 4a$.

$\Rightarrow y_1 = \frac{4a}{2y} = \frac{2a}{y}$

$\Rightarrow y_2 = -\frac{2a}{y^2} \cdot y_1 = \frac{-2a}{y^2} \times \frac{2a}{y}$

$\Rightarrow y_2 = \frac{-4a^2}{y^3}$ Subst. in (1)

$\Rightarrow \rho = \frac{\left[1+\frac{4a^2}{y^2}\right]^{3/2}}{\frac{-4a^2}{y^3}}$

$$\Rightarrow \rho = \frac{[y^2+4a^2]^{3/2}}{y^3} \times \frac{y^3}{-4a^2}$$

$$\Rightarrow \rho = \frac{[4ax+4a^2]^{3/2}}{-4a \cdot a}$$

$(2^2)^{1/2} a^{1/2}$

$\frac{2\sqrt{a}}{\sqrt{a}}$

$$\Rightarrow \rho = \frac{(4a)^{3/2}[x+a]^{3/2}}{-(4a)\,a}$$

$$\Rightarrow \rho = \frac{(4a)^{3/2-1}[x+a]^{3/2}}{-a}$$

$$\Rightarrow \rho = \frac{(4a)^{1/2}\{x+a\}^{3/2}}{-a} = \frac{2\sqrt{a}.\,[x+a]^{3/2}}{-a}$$

$$\Rightarrow \rho = \frac{+2}{\sqrt{a}}[x+a]^{3/2}.$$

3. find the radius of curvature at point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the curve. $x^3+y^3 = 3axy$.

$(uv)' = uv' + vu'$

Sol Given $x^3+y^3 = 3axy$.

D. w. r. to x.

$$\Rightarrow \cancel{3}x^2 + \cancel{3}y^2 . y_1 = \cancel{3}a[x y_1 + y]$$

$$\Rightarrow x^2 + y^2 y_1 = axy_1 + ay.$$

$$\Rightarrow y^2 y_1 - axy_1 = ay - x^2$$

$$\Rightarrow y_1 . [y^2 - ax] = ay - x^2$$

$$\Rightarrow y_1 = \frac{ay - x^2}{y^2 - ax}.$$

$\left(\frac{u}{v}\right)' = \frac{vu' - uv'}{v^2}$

$$\Rightarrow y_1\left(\tfrac{3a}{2}, \tfrac{3a}{2}\right) = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1 \Rightarrow \boxed{y_1 = -1}$$

$\sqrt{2}$

$$y_2 = \frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1}{(y^2-ax)^2}$$

$$y_2\left(\frac{3a}{2},\frac{3a}{2}\right) = \frac{\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)\left(-a-\frac{6a}{2}\right)-\left(\frac{3a^2}{2}-\frac{9a^2}{4}\right)\left(-\frac{6a}{2}\dots\right)}{\left(\frac{9a^2}{4}-\frac{3a^2}{2}\right)^2}$$

$$y_2 = \frac{\left(\frac{9a^2-6a^2}{4}\right)\left(\frac{-2a-6a}{2}\right)-\left(\frac{6a^2-9a^2}{4}\right)\left(\frac{-6a-2a}{2}\right)}{\left(\frac{9a^2-6a^2}{4}\right)^2}$$

$$y_2 = \frac{\frac{3a^2}{4}\times\left(-\frac{\cancel{8}^{4}a}{\cancel{2}}\right)-\left(\frac{-3a^2}{4}\right)\left(-\frac{\cancel{8}^{4}a}{\cancel{2}}\right)}{\left(\frac{3a^2}{4}\right)^2}$$

$$y_2 = \frac{-12a^3-12a^3}{\cancel{4}}\times\frac{16^{4}}{9a^4}$$

$$y_2 = \frac{-\cancel{24}^{8}a^3\times 4}{\cancel{9}_{3}a^4}$$

$$y_2 = -\frac{32}{3a}$$

Radius of curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$$\rho = \frac{[1+1]^{3/2}}{\frac{-32}{3a}}$$

$$\rho = \frac{2^{3/2}\times 3a}{-32}$$

$$\Rightarrow \rho = \frac{\sqrt{8}\times 3a}{-32} = \frac{\cancel{2}\sqrt{2}\times 3a}{\cancel{-32}_{16}} = \frac{\sqrt{2}\times 3a}{8\times 2} = \frac{3a}{8\sqrt{2}}$$

4. Find the radius of curvature of the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ at $\left[\frac{a}{4},\frac{a}{4}\right]$

Sol $\sqrt{x}+\sqrt{y}=\sqrt{a}$.

Diff. w.r.t. x.

$$\frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}y_1=0.$$

$$\Rightarrow \frac{y_1}{\cancel{2}\sqrt{y}}=-\frac{1}{\cancel{2}\sqrt{x}}$$

$$\Rightarrow y_1=-\sqrt{\frac{y}{x}}.$$

$$\Rightarrow y_1\left(\frac{a}{4},\frac{a}{4}\right)=-\sqrt{\frac{a/4}{a/4}}=-\sqrt{1}=-1$$

$$\Rightarrow y_2=\frac{-1}{2\sqrt{y/x}}\left(\frac{xy_1-y}{x^2}\right)$$

$$\Rightarrow y_2=\frac{-1}{2(-y_1)}\left(\frac{-x-y}{x^2}\right)$$

$$\Rightarrow y_2\left(\frac{a}{4},\frac{a}{4}\right)=\frac{-1}{2\times1}\left(\frac{-\frac{a}{4}-\frac{a}{4}}{\frac{a^2}{16}}\right)$$

$$\Rightarrow y_2=\frac{-1}{\cancel{2}}\left(-\frac{\cancel{2}\cancel{a}}{\cancel{4}}\times\frac{\cancel{16}^{\cancel{8}\,4}}{a^{\cancel{2}}}\right)$$

$$\Rightarrow y_2=\frac{4}{a}.$$

Radius of Curvature $\rho=\frac{[1+y_1^2]^{3/2}}{y_2}$

$$\Rightarrow \rho=\frac{[1+1]^{3/2}}{\frac{4}{a}}=2^{3/2}\times\frac{a}{4}=\frac{\cancel{2}\sqrt{2}\times a}{\cancel{4}2}=\frac{a}{\sqrt{2}}$$

3. Find the radius of curvature at any point 2

(i) Astroid. $x^{2/3} + y^{2/3} = a^{2/3}$

Sol:- The parametric equations of astroid.

$x = a\cos^3\theta$ & $y = a\sin^3\theta$.

$\Rightarrow$ Diff. w.r.t. θ

$\Rightarrow \frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta)$

$\Rightarrow \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$\Rightarrow y_1 = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^{\cancel{2}}\theta\,\cancel{\cos\theta}}{-3a\cos^{\cancel{2}}\theta\,\cancel{\sin\theta}} = -\tan\theta$

$\Rightarrow y_1 = -\tan\theta$

$\Rightarrow y_2 = -\sec^2\theta \cdot \frac{d\theta}{dx}$

$\Rightarrow y_2 = -\sec^2\theta \cdot \frac{1}{-3a\cos^2\theta\sin\theta}$

$\Rightarrow y_2 = \frac{\sec^4\theta}{3a\sin\theta}$

Radius of curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$\Rightarrow \rho = \frac{[1+\tan^2\theta]^{3/2}}{\frac{\sec^4\theta}{3a\sin\theta}} = \frac{(\sec^{\cancel{2}}\theta)^{3/\cancel{2}} \times 3a\sin\theta}{\sec^4\theta}$

$\rho = 3a\sin\theta\cos\theta$

$\Rightarrow \rho = \frac{3a\sin2\theta}{2}$

6. Prove that the Radius of curvature at any point on the cycloid $x = a(\theta + \sin\theta)$ & $y = a(1-\cos\theta)$ is $4a\cos\theta/2$.

Sol

$$x = a(\theta+\sin\theta) \quad \& \quad y = a(1-\cos\theta)$$

Diff. w.r.t. θ

$$\Rightarrow \frac{dx}{d\theta} = a(1+\cos\theta) \quad \& \quad \frac{dy}{d\theta} = a(\theta+\sin\theta)$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cancel{a}(\theta+\sin\theta)}{\cancel{a}(1+\cos\theta)}.$$

$$\Rightarrow y_1 = \frac{\sin\theta}{1+\cos\theta} = \frac{\cancel{2}\sin\theta/2\,\cancel{\cos\theta/2}}{\cancel{2}\cos^{\cancel{2}}\theta/2}$$

$$\Rightarrow y_1 = \tan\theta/2$$

$$\Rightarrow y_2 = \sec^2\theta/2 \times \frac{1}{2}\cdot\frac{d\theta}{dx}$$

$$\Rightarrow y_2 = \frac{1}{2}\sec^2\theta/2 \cdot \frac{1}{a(1+\cos\theta)}$$

$$\Rightarrow y_2 = \frac{\sec^2\theta/2}{2a\cdot 2\cos^2\theta/2}$$

$$\Rightarrow y_2 = \frac{\sec^4\theta/2}{4a}.$$

Radius of curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$.

$$\Rightarrow \rho = \frac{[1+\tan^2\theta/2]^{3/2}}{\frac{\sec^4\theta/2}{4a}} = \frac{(\sec^2\theta/2)^{3/2}\times 4a}{\sec^4\theta/2}$$

$$\Rightarrow \rho = 4a\cos\theta/2$$

7. Find the Radius of Curvature $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$.

Sol $x = a(\cos t + t\sin t)$; $y = a(\sin t - t\cos t)$

D. w.r.to t

$\Rightarrow \frac{dx}{dt} = a.(-\cancel{\sin t} + t.\cos t + \cancel{\sin t})$

$\Rightarrow \frac{dx}{dt} = at\cos t$

$\Rightarrow \frac{dy}{dt} = a.(\cancel{\cos t} - t.(-\sin t) - \cancel{\cos t})$

$\Rightarrow \frac{dy}{dt} = at\sin t.$

$y_1 = \frac{dy/dt}{dx/dt} = \frac{at\sin t}{at\cos t} = \tan t$

$y_2 = \sec^2 t \frac{dt}{dx} = \sec^2 t . \frac{1}{at\cos t} = \frac{\sec^3 t}{at}$

Radius of Curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$\Rightarrow \rho = \frac{[1+\tan^2 t]^{3/2}}{\frac{\sec^3 t}{at}} = \frac{\cancel{\sec^3 t} \times at}{\cancel{\sec^3 t}}$

$\Rightarrow \rho = at$

8. find the Radius of Curvature at a point $(at^2, 2at)$ on the Curve $y^2 = 4ax$.

Sol $y^2 = 4ax$.

Diff. w.r.to x.

$\Rightarrow 2yy_1 = 4a$

$\Rightarrow y_1 = \frac{\cancel{4}^2 a}{\cancel{2}y} = \frac{2a}{y}$

$\Rightarrow y_1 = \frac{2a}{y}$

$\Rightarrow y_1(at^2, 2at) = \frac{2a}{2at} = \frac{1}{t}$

$\Rightarrow y_1 = \frac{1}{t}$

$\Rightarrow y_2 = \frac{y(0) - y_1 2a}{y^2} = \frac{-2a}{y^2} y_1$

$= \frac{-2a}{4a^2t^2} \times \frac{1}{t}$

$$\boxed{\begin{aligned} &\Rightarrow y_2 = \frac{-2a\left(\frac{2a}{y}\right)}{y^2} = \frac{-4a^2}{y^3} \\ &\Rightarrow y_2(at^2, 2at) = \frac{-4a^2}{8a^3t^3} \end{aligned}} = \frac{-1}{2at^3}$$

Radius of Curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$\Rightarrow \rho = \frac{\left[1+\frac{1}{t^2}\right]^{3/2}}{\frac{-1}{2at^3}} = \frac{(t^2+1)^{3/2}}{t^3} \times -2at^3$

$\Rightarrow \rho = -2a(t^2+1)^{3/2}$

$\Rightarrow \rho = 2a(t^2+1)^{3/2}$

Note:- Radius of Curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2} \Rightarrow$ D.w.r.to x

If $y_1 = \infty$ then R.C $= \frac{[1+x_1^2]^{3/2}}{x_2} \Rightarrow$ D.w.r.to y.

9. Find the Radius of Curvature at a point $(-2, 0)$ on the Curve $y^2 = x^3 + 8$.

Sol. $y^2 = x^3 + 8$

→ D.w.r. to x

→ $2y\,y_1 = 3x^2$

→ $y_1 = \frac{3x^2}{2y}$ ⇒ $y_{1(-2,0)} = \frac{3\times 4}{2\times 0} = \infty$

→ D.w.r. to y

→ $2y = 3x^2 x_1$

→ $x_1 = \frac{2y}{3x^2}$

⇒ $x_{1(-2,0)} = 0$

⇒ $x_2 = \frac{2}{3}\left[\frac{x^2(1) - y(2x\,x_1)}{x^4}\right]$

⇒ $x_{2(-2,0)} = \frac{2}{3}\left[\frac{4 - 0}{16}\right] = \frac{1}{6}$

Radius of Curvature $\rho = \frac{[1 + x_1^2]^{3/2}}{x_2}$

$y_1 = \frac{dy}{dx}$

⇒ $\rho = \frac{[1+0]^{3/2}}{\frac{1}{6}} = 6$

$x_1 = \frac{dx}{dy}$

(7) 10. Find the radius of Curvature at a point $(\sqrt{2}, \sqrt{2})$ on the Curve $x^2 + y^2 = 4$ D.w.r. to y.

Ans = +2

Sol $x^2 + y^2 = 4$

D.w.r. to y

⇒ $2x\,x_1 + 2y = 0$.

$\Rightarrow y_1 = -\frac{2y}{2x} = -\frac{x}{y}$

$\Rightarrow y_1(\sqrt{2},\sqrt{2}) = -1$

$\Rightarrow y_2 = -\left[\frac{y(x_1) - x}{y^2}\right] = \frac{x - y(-1)}{y^2}$

$\Rightarrow y_2 = \frac{x+y}{y^2} \Rightarrow y_2(\sqrt{2},\sqrt{2}) = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$

$\therefore$ Radius of curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$\Rightarrow \rho = \frac{[1+(-1)^2]^{3/2}}{\sqrt{2}} = \frac{(2)^{3/2}}{\sqrt{2}} = \frac{2\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}}$

$\Rightarrow \rho = 2$

$\therefore$ The Radius of Curvature

$\boxed{\rho = 2}$

Q11. Find the radius of curvature at any point (r,θ) on the curve $r = a(1-\cos\theta)$

Sol $r = a(1-\cos\theta)$

Radius of curvature $\rho = \frac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - r_1 r_2}$

D.w.r. to θ.

$\frac{dr}{d\theta} = a(\sin\theta)$

$r_1 = a\sin\theta$

$r_2 = a\cos\theta$

$\Rightarrow \rho = \frac{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(1-\cos\theta)(a\cos\theta)}$

$\Rightarrow \rho = \frac{a^{3}[1+\cos^2\theta - 2\cos\theta + \sin^2\theta]^{3/2}}{\cancel{a^2}[1+\cos^2\theta - 2\cos\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta]}$

$\frac{a(2-2\cos\theta)^{3/2}}{} = \frac{2^{3/2}a(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}$

$$\Rightarrow \rho = \frac{2\sqrt{2}}{3} a (1-\cos\theta)^{3/2 - 1}$$

$$\Rightarrow \rho = \frac{2\sqrt{2}}{3} a (1-\cos\theta)^{1/2}$$

$$\Rightarrow \rho = \frac{2\sqrt{2}a}{3} \sqrt{2\sin^2 \theta/2}$$

$$\Rightarrow \rho = \frac{2.\sqrt{2}.a\sqrt{2}.\sin\theta/2}{3}$$

$$\Rightarrow \rho = \frac{4a}{3} \sin\theta/2$$

12. Find the radius of curvature at a point (r,θ) on the curve $r^n = a^n \cos n\theta$.

Sol $r^n = a^n \cos n\theta$.

Radius of curvature $\rho = \frac{[r^2 + r_1^2]^{3/2}}{r^2 + 2r_1^2 - 2rr_2}$

Take log on b.s.

$\log r^n = \log a^n \cos n\theta$

$n \log r = n \log a + \log \cos n\theta$.

D. w.r. to θ

$\cancel{n}. \frac{1}{r}. r_1 = \frac{1}{\cos n\theta} (-\sin n\theta)\cancel{n}$.

$\Rightarrow r_1 = -r. \tan n\theta. \Rightarrow r_{1(r,\theta)} = -r \tan n\theta$.

D. w.r. to θ.

$\Rightarrow r_2 = -[r \sec^2 n\theta . n. + \tan n\theta . r_1]$

$\Rightarrow r_2 = -[rn \sec^2 n\theta] + r \tan^2 n\theta$

$\Rightarrow r_2 = r[\tan^2 n\theta - n\sec^2 n\theta]$.

$$\Rightarrow \rho = \frac{[r^2 + r^2\tan^2 n\theta]^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2[\tan^2 n\theta - n\sec^2 n\theta]}$$

$$\Rightarrow \rho = \frac{r^3[1+\tan^2 n\theta]^{3/2}}{\cancel{r^2}[1+2\tan^2 n\theta - \tan^2 n\theta + n\sec^2 n\theta]}$$

$$\Rightarrow \rho = \frac{r\sec^3 n\theta}{[1+\tan^2 n\theta + n\sec^2 n\theta]}$$

$$\Rightarrow \rho = \frac{r\sec^3 n\theta}{\sec^2 n\theta + n\sec^2 n\theta}$$

$$\Rightarrow \rho = \frac{r\sec^{\cancel{3}} n\theta}{\cancel{\sec^2} n\theta(1+n)}$$

$$\frac{r^n}{a^n} = \cos n\theta$$

$$\sec n\theta = \frac{a^n}{r^n}$$

$$\Rightarrow \rho = \frac{r\sec n\theta}{1+n}.$$

$$\Rightarrow \rho = \frac{r \cdot \frac{a^n}{r^n}}{1+n}.$$

$$\Rightarrow \rho = \frac{a^n}{(1+n)r^{n-1}}$$

$$\Rightarrow \rho = \frac{a^n}{(1+n)r^{n-1}}$$

$(r^2 + r_1^2)^{3/2}$

$r^2 + 2r_1^2 - rr_2$

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Note: 1 The centre of curvature at any point $P(x,y)$ on the curve $y = f(x)$ the centre of curvature at any point $P(x,y)$ is given as

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\bar{Y} = y + \frac{(1+y_1^2)}{y_2}$$

2. The equation of circle of curvature at any point $P(x,y)$ is given by $(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$ where $(\bar{x}, \bar{y})$ be the centre of circle and ρ.

ρ = radius of the circle.

1. Find the equation of circle of curvature of the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at point $\left(\frac{a}{4}, \frac{a}{4}\right)$

Sol $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

diff. w.r.t x.

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} y_1 = 0$$

$$\Rightarrow \frac{y_1}{\cancel{2}\sqrt{y}} = -\frac{1}{\cancel{2}\sqrt{x}}$$

$$\Rightarrow y_1 = -\sqrt{y}/\sqrt{x} = -\sqrt{\frac{y}{x}} \Rightarrow y_1\left(\frac{a}{4}, \frac{a}{4}\right) = -1$$

$$\Rightarrow y_2 = \frac{-1}{2\sqrt{y/x}}\left[\frac{x y_1 \mp y}{x^2}\right]$$

$$\Rightarrow y_2 = \frac{-1}{2\sqrt{y/x}}\left[\frac{-x \mp y}{x^2}\right]$$

$$y_2 = \frac{-1}{-2y_1}\left[-\frac{x-y}{x^2}\right] = -\frac{1}{2}\left[\frac{-x-y}{x^2}\right]$$

$$y_2\left(\frac{a}{4},\frac{a}{4}\right) = -\frac{1}{2}\left[\frac{-\frac{a}{4}-\frac{a}{4}}{\frac{a^2}{16}}\right] = -\frac{1}{\cancel{2}}\left[-\frac{\cancel{2}a}{\cancel{4}}\times\frac{\overset{4}{\cancel{16}}}{a^2}\right]$$

$$y_2\left(\frac{a}{4},\frac{a}{4}\right) = \frac{4}{a}.$$

The circle of curvature of the curve is

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2.$$

$$\Rightarrow \bar{x} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\Rightarrow \bar{x} = x - \frac{(-1)(1+1)}{\frac{4}{a}} = x + \frac{\cancel{2}a}{\cancel{4}2} = x + \frac{a}{2} = \frac{a}{4}+\frac{a}{2} = \frac{3a}{4}$$

$$\Rightarrow \bar{y} = y + \frac{(1+y_1^2)}{y_2} = y + \frac{(1+1)}{\frac{4}{a}} = y + \frac{a}{2} = \frac{a}{4}+\frac{a}{2} = \frac{3a}{4}.$$

Centre $= \left(\frac{3a}{4}, \frac{3a}{4}\right)$

$$\Rightarrow \cancel{\left(\cancel{x} - \cancel{x} - \frac{a}{2}\right)^2 + \left(\cancel{y} - \cancel{y} - \frac{a}{2}\right)^2 = \rho^2}$$

$$\Rightarrow \cancel{\rho^2 = \frac{a^2}{4} + \frac{a^2}{4} = \frac{\cancel{2}a^2}{\cancel{4}2} = \frac{a^2}{2}}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{4/a} = (2)^{3/2}\frac{a}{4} = \sqrt{8}\,\frac{a}{4} = \cancel{2}\sqrt{2}\,\frac{a}{\cancel{4}2} = \sqrt{2}\cdot\frac{a}{2} = \frac{\cancel{2}\,a}{\cancel{2}\cdot\sqrt{2}}$$

$$\Rightarrow \rho^2 = \frac{a^2}{2}$$

$$\Rightarrow \rho = \sqrt{\frac{a^2}{2}} = \frac{a}{\sqrt{2}}$$

$$\therefore \rho = \frac{a}{\sqrt{2}}$$

Equation of cle of curvature

$$\Rightarrow (x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\Rightarrow \left(x-\frac{3a}{4}\right)^2 + \left(y-\frac{3a}{4}\right)^2 = \left(\frac{a}{\sqrt{2}}\right)^2$$

$$\Rightarrow \frac{(4x-3a)^2}{16} + \frac{(4y-3a)^2}{16} = \frac{a^2}{2}$$

$$\Rightarrow (4x-3a)^2 + (4y-3a)^2 = 8a^2$$

2. Find the equation of circle of curvature of the curve $x^3+y^3 = 3xy$ at $(3/2, 3/2)$

Sol $x^3+y^3 = 3xy$

Diff. w.r.t x

$$\Rightarrow 3x^2 + 3y^2 \cdot y_1 = 3(x \cdot y_1 + y)$$

$$\Rightarrow 3x^2 + 3y^2 y_1 = 3xy_1 + 3y$$

$$\Rightarrow 3y^2 y_1 - 3xy_1 = 3y - 3x^2$$

$$\Rightarrow \cancel{3}(y^2-x)y_1 = \cancel{3}(y-x^2)$$

$$\Rightarrow y_1 = \frac{y-x^2}{y^2-x}$$

$$\Rightarrow y_1\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{\frac{3}{2}-\frac{9}{4}}{\frac{9}{4}-\frac{3}{2}} = \frac{6-9}{9-6} = \frac{-3}{3} = -1$$

$$\Rightarrow y_2 = \frac{(y^2-x)(y_1-2x) - (y-x^2)(2yy_1-1)}{(y^2-x)^2}$$

$$\Rightarrow y_2 = \frac{(y^2-x)(-1-2x) - (y-x^2)(-2y-1)}{(y^2-x)^2} \quad y_2 = y$$

$$\Rightarrow y_2 = \frac{(-1-2x)[y^2-x-y+x^2]}{(y^2-x)^2}$$

$$\Rightarrow y_2\left(\frac{3}{2}, \frac{3}{2}\right) = \frac{(-1-3)\left[\frac{9}{4}-\frac{3}{2}-\frac{3}{2}+\frac{9}{4}\right]}{\left(\frac{9}{4}-\frac{3}{2}\right)^2}$$

$$y_2 = -\frac{(18-12)}{9} \times 16 = -\frac{\cancel{6} \times 16}{\cancel{9}3} = -\frac{32}{3}$$

Radius of curvature $\rho = \frac{[1+y_1^2]^{3/2}}{y_2}$

$$\Rightarrow \rho = \frac{[1+1]^{3/2}}{-\frac{32}{3}} = \frac{2^{3/2} \times 3}{-32} = \frac{\cancel{2}\sqrt{2} \times 3}{-\frac{\cancel{32}}{16}} = \frac{3\sqrt{2}}{16} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3}{8\sqrt{2}}$$

Let $(\bar{x}, \bar{y})$ be the centre of circle of curvature

$$\bar{x} = x - y_1 \frac{(1+y_1^2)}{y_2}$$

$$= \frac{3}{2} + 1 \cdot \frac{(1+1)}{-\frac{32}{3}} = \frac{3}{2} - \frac{\cancel{6}3}{\frac{\cancel{32}}{16}} = \frac{24-3}{16} = \frac{21}{16}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2}$$

$$= \frac{3}{2} + \frac{(1+1)}{-\frac{32}{3}} = \frac{3}{\cancel{2}} + \frac{\cancel{6}3}{-\frac{\cancel{32}}{16}} = \frac{3}{2} - \frac{3}{16} = \frac{24-3}{16}$$

$$= \frac{21}{16}$$

Centre $= \left(\frac{21}{16}, \frac{21}{16}\right)$.

Equation of circle of curvature

$$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\Rightarrow \left(x - \frac{21}{16}\right)^2 + \left(y - \frac{21}{16}\right)^2 = \left(\frac{3}{8\sqrt{2}}\right)^2$$

$$\Rightarrow \frac{(16x-21)^2}{16^2} + \frac{(16y-21)^2}{16^2} = \frac{9}{64 \times 2}$$

$$\Rightarrow (16x-21)^2 + (16y-21)^2 = \frac{\cancel{256}^{2} \times 9}{\cancel{128}} = 18$$

$$\Rightarrow (16x-21)^2 + (16y-21)^2 = 18.$$

$$\begin{array}{r} 64 \\ \times 2 \\ \hline 128 \end{array} \qquad \begin{array}{r} 16 \\ \times 16 \\ \hline 96 \\ 160 \\ \hline 256 \end{array} \qquad \begin{array}{r} 128 \\ \times 2 \\ \hline 256 \end{array}$$

2/10/2011

1. Show that the Circle of Curvature at Origin of parabola $y = mx + \frac{x^2}{a}$ is $x^2 + y^2 = a(1+m^2)(y-mx)$

Sol $y = mx + \frac{x^2}{a}$ at $(0,0)$.

Diff. w.r.to 'x'.

$$y_1 = m + \frac{2x}{a}$$

$$\boxed{y_1(0,0) = m}$$

$$y_2 = \frac{2}{a}$$

$$\boxed{y_2(0,0) = \frac{2}{a}}$$

Radius of Curvature $\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$

$$\rho = \frac{(1+m^2)^{3/2}}{2/a} = \frac{a}{2}(1+m^2)^{3/2}$$

Let $(\bar{x}, \bar{y})$ be the Centre of Curvature.

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = a - \frac{m(1+m^2)}{\cancel{2/a}} = \frac{-am(1+m^2)}{2}$$

$$\bar{y} = y + \frac{(1+y_1^2)}{y_2} = 0 + \frac{1+m^2}{2/a} = \frac{a}{2}(1+m^2)$$

$\therefore$ Centre $\left(-\frac{ma}{2}(1+m^2); \frac{a}{2}(1+m^2)\right)$

Eqn of cle of curvature.

$$\Rightarrow (x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$$

$$\Rightarrow \left(x + \frac{ma}{2}(1+m^2)\right)^2 + \left(y - \frac{a}{2}(1+m^2)\right)^2 = \left(\frac{a}{2}\right)^2(1+m^2)^3$$

$$\Rightarrow \left[x^2 + \frac{m^2a^2}{4}(1+m^2)^2 + max(1+m^2)\right] +$$

$$\left[y^2 + \frac{a^2}{4}(1+m^2)^2 - ay(1+m^2)\right] = \frac{a^2}{4}(1+m^2)^3$$

$$\Rightarrow x^2+y^2 = \frac{a^2}{4}(1+m^2)^3 - \frac{m^2a^2}{4}(1+m^2)^2 - max(1+m^2)$$
$$- \frac{a^2}{4}(1+m^2)^2 + ay(1+m^2)$$

$$\Rightarrow x^2+y^2 = a(1+m^2)\left[\frac{a}{4}(1+m^2)^2 - \frac{m^2a}{4}(1+m^2) - mx - \frac{a}{4}(1+m^2) + y\right]$$

$$\Rightarrow x^2+y^2 = a(1+m^2)\left[(y-mx) + \frac{a}{4}(1+m^2)\{\cancel{1}+m^2-m^2-1\}\right]$$

$$\Rightarrow x^2+y^2 = a(1+m^2)\left[(y-mx)+0\right]$$

$$\therefore x^2+y^2 = \underline{a(1+m^2)(y-mx)}.$$

2. £

$b\bar{y} = -(a^2-b^2)\sin^3\theta$.

$(b\bar{y})^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta \longrightarrow (2)$

Add (1) + (2)

$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}(\cos^2\theta + \sin^2\theta)$

$(a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3}$.

∴ The locus of Centre of Curvature is

$\underline{(ax)^{2/3} + (by)^{2/3}} = (a^2-b^2)^{2/3}$.

(1). Find the Envelope $x\cos\alpha + y\sin\alpha = p$

Ans $x^2+y^2=p^2$

(2) $x\tan\alpha + y\sec\alpha = 5 \rightarrow (1)$ α p~

(1) - (2)

$y^2 - x^2 = 25$

$x\sec^2\alpha + y\sec\alpha\tan\alpha = 0$

$x\sec\alpha + y\tan\alpha = 5 \rightarrow (2)$

(3) $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$ α v p.

$(1)^2 + (2)^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

14/10/2011 R. G. Envelopes (5+2)

One parameter family of curves :-

Let $f(x, y, \alpha)$ be a function of 3 variables. then the equation of $f(x, y, \alpha) = 0$ represents a curve corresponding to each particular value of α. the collection of all this curves is called "One parameter family of curves". and α is called parameter.

Eg:- ① $y = mx + c$. represents a family of curves for different values of m. here m is parameter

② $x^2 + y^2 = a^2$. represents a family of curves for different values of 'a' here a is parameter.

1. The Envelope of one parameter family of curves is the locus of the limiting position of the point of intersection of any two curves. of the family when one of them tends to coincide with other curve which is fixed.

Working rule to find envelopes

Method-1 :- ① Let the family of curves be

$f(x, y, \alpha) = 0 \longrightarrow ①$

② partially differentiate w.r.to α i.e.,

$$\frac{\partial}{\partial \alpha} f(x, y, \alpha) = 0 \longrightarrow ②$$

③ eliminate α by using ① & ② then we get envelope of the given curve

<u>Method 2:-</u>

1. If the equation of the family of the curves be quadratic eqn. in parameter α. i.e., $a\alpha^2 + b\alpha + c = 0$ then the envelope of family of the curves be $b^2 - 4ac = 0$.

<u>Method 3:-</u>

If the eqn of family of the curves be $A\cos\alpha + B\sin\alpha = C$ then envelope is $A^2 + B^2 = C^2$.

*1. find the envelope of family of curves. where 'θ' is parameter.

$$x\cos^3\theta + y\sin^3\theta = a$$

<u>Sol</u> $x\cos^3\theta + y\sin^3\theta = a \longrightarrow ①$

P.D w.r.to θ

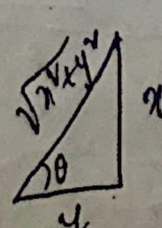
$$3x\cos^2\theta \cdot (-\sin\theta) + 3y\sin^2\theta(\cos\theta) = 0$$

$$-\cancel{3}x\cos^{\cancel{2}}\theta\,\cancel{\sin\theta} + \cancel{3}y\sin^{\cancel{2}}\theta\,\cancel{\cos\theta} = 0$$

$$-x\cos\theta + y.\sin\theta = 0$$

$$x\cos\theta = y\sin\theta$$

$$\frac{\sin\theta}{\cos\theta} = \frac{x}{y}.$$



$$\tan\theta = x/y \rightarrow ②$$

$\cos\theta = \dfrac{y}{\sqrt{x^2+y^2}}$; $\sin\theta = \dfrac{x}{\sqrt{x^2+y^2}}$ subsb. in ①

$$\Rightarrow x\,\frac{y^3}{(\sqrt{x^2+y^2})^3} + y\,\frac{x^3}{(\sqrt{x^2+y^2})^3} = a.$$

$$\Rightarrow xy\left[\frac{x^2+y^2}{(x^2+y^2)^{3/2}}\right] = a$$

$$\Rightarrow \frac{xy}{(x^2+y^2)^{1/2}} = a$$

$$\Rightarrow xy = \underline{a\sqrt{x^2+y^2}}.$$

2. find the envelope of family of curves normal to ellipse is $ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2$. where '$\theta$' is parameter.

Sol: $ax\sec\theta - by\operatorname{cosec}\theta = a^2-b^2 \rightarrow$ ①.

P.D. w.r.to θ

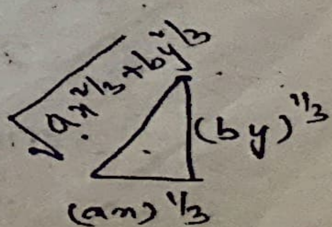
$$\Rightarrow ax\sec\theta\tan\theta + by\operatorname{cosec}\theta\cot\theta = 0.$$

$$\Rightarrow ax\sec\theta\tan\theta = -by\operatorname{cosec}\theta\cot\theta.$$

$$\Rightarrow ax\,\frac{1}{\cos\theta}\,\frac{\sin\theta}{\cos\theta} = -by\,\frac{1}{\sin\theta}\,\frac{\cos\theta}{\sin\theta}$$

$$\Rightarrow \frac{ax\sin\theta}{\cos^2\theta} = -by\,\frac{\cos\theta}{\sin^2\theta}.$$

$$\Rightarrow ax\sin^3\theta = -by\cos^3\theta.$$



$$\Rightarrow \frac{\sin^3\theta}{\cos^3\theta} = \frac{-by}{ax}$$

$$\Rightarrow \tan^3\theta = -\frac{by}{ax}.$$

$$\Rightarrow \tan\theta = \left(-\frac{by}{ax}\right)^{1/3} = \frac{-(by)^{1/3}}{ax^{1/3}}$$

$\cos\theta$ =
$\sec\theta = \dfrac{\sqrt{ax^{2/3}+by^{2/3}}}{ax^{1/3}}$; $-\operatorname{cosec}\theta = \dfrac{\sqrt{ax^{2/3}+by^{2/3}}}{by^{1/3}}$

IV quad. Subst in ①.

$$\Rightarrow ax \cdot \frac{\sqrt{ax^{2/3}+by^{2/3}}}{ax^{1/3}} + by\,\frac{\sqrt{ax^{2/3}+by^{2/3}}}{by^{1/3}} = a^2-b^2$$

$$\Rightarrow \sqrt{ax^{2/3}+by^{2/3}}\left[ax^{1-1/3} + by^{1-1/3}\right] = a^2-b^2$$

$$\Rightarrow \sqrt{ax^{2/3}+by^{2/3}}\,(ax^{2/3}+by^{2/3}) = a^2-b^2.$$

$$\Rightarrow (ax^{2/3}+by^{2/3})^{1/2+1} = a^2-b^2$$

$$\Rightarrow (ax^{2/3}+by^{2/3})^{3/2} = a^2-b^2$$

$$\therefore\ ax^{2/3}+by^{2/3} = (a^2-b^2)^{2/3}.$$

3. Consider the Evolute of the curve as envelope of its normal. find the envelope of the parabola $y^2 = 4ax$.

Sol Eqn of normal to parabola is $y = mx - 2am - am^3$ ①

P.D. w.r.to m

$\Rightarrow 0 = x - 2a - 3am^2.$

$\Rightarrow 3am^2 = x-2a$

$\Rightarrow m^2 = \dfrac{x-2a}{3a}.$

$y = mx - am^2 - 2am$

$y = mx - 2am - am^3$

from ① $y = m(x-2a) - am^3.$

Sq. on b.s.

$$y^2 = [m^2(x-2a) - am^3]^2$$

$$\Rightarrow y^2 = m^2(x-2a)^2 + a^2m^6 - 2am^4(x-2a).$$

$$\Rightarrow y^2 = \frac{(x-2a)}{3a}(x-2a)^2 + a^2\left(\frac{x-2a}{3a}\right)^3 - 2a\left(\frac{x-2a}{3a}\right)^2(x-2a)$$

$$\Rightarrow y^2 = \frac{(x-2a)^3}{3a} + \frac{\cancel{a^2}(x-2a)^3}{27a^{\cancel{3}}} - \frac{2\cancel{a}(x-2a)^3}{9a^{\cancel{2}}}$$

$$\Rightarrow y^2 = \frac{9(x-2a)^3 + (x-2a)^3 - 6(x-2a)^3}{27a}.$$

$$\Rightarrow 27ay^2 = 4(x-2a)^3$$

4. Find the envelope of $y = mx + \sqrt{a^2m^2+b^2}$ where 'm' is parameter.

Sol

$$y = mx + \sqrt{a^2m^2+b^2}.$$

$$\Rightarrow y - mx = \sqrt{a^2m^2+b^2}$$

sq. on both sides.

$$\Rightarrow (y-mx)^2 = (\sqrt{a^2m^2+b^2})^2.$$

$$\Rightarrow y^2 + m^2x^2 - 2mxy = a^2m^2 + b^2$$

$$\Rightarrow y^2 + m^2x^2 - a^2m^2 - 2mxy + y^2 - b^2 = 0.$$

$$\Rightarrow m^2x^2 - a^2m^2 - 2mxy + y^2 - b^2 = 0.$$

$$\Rightarrow m^2(x^2-a^2) - (2xy)m + (y^2-b^2) = 0.$$

this is Q. eqn in terms of envelope is

$$b^2 - 4ac = 0.$$

$$\Rightarrow (-2xy)^2 - 4(x^2-a^2)(y^2-b^2) = 0.$$

$$\Rightarrow \cancel{4}x^2y^2 - \cancel{4}[x^2y^2 - x^2b^2 - a^2y^2 + a^2b^2] = 0.$$

$$\Rightarrow \cancel{x^2y^2} - \cancel{x^2y^2} + x^2b^2 + a^2y^2 - a^2b^2 = 0$$

$\Rightarrow x^2b^2 + y^2a^2 = a^2b^2$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

5. Find envelope of $y = mx + \frac{a}{m}$ where 'm' is parameter

Sol Given $y = mx + \frac{a}{m} \rightarrow (1)$

~~R.D~~

$my = m^2x + a.$

~~R.D. w.r.to m~~

~~Sq. on b.s.~~

$m^2x - my + a = 0.$

this is Q.E in terms of m. is $b^2 - 4ac = 0$.

$\Rightarrow y^2 - 4xa = 0.$

$\Rightarrow y^2 = 4ax.$

6. Find the envelope of $y = mx + \sqrt{1+m^2}$ where 'm' is parameter

Sol $y = mx + \sqrt{1+m^2} \Rightarrow y - mx = \sqrt{1+m^2}.$

Sq. on. both sides

$\Rightarrow y^2 + m^2x^2 + 1 + m^2 + 2mxy = 0.$

$\Rightarrow \cancel{m^2(x^2+1) - y^2 + 1 = 0.}$

$\Rightarrow m^2(x^2-1) + (2xy)m + (y^2-1) = 0.$

this is Q.E in terms of m is $b^2 - 4ac = 0$

$\Rightarrow (2xy)^2 - 4(x^2-1)(y^2-1) = 0.$

$\Rightarrow \cancel{4}x^2y^2 - \cancel{4}(x^2y^2 - x^2 - y^2 + 1) = 0$

$\Rightarrow \cancel{x^2y^2} - \cancel{x^2y^2} + x^2 + y^2 - 1 = 0$

$\Rightarrow \underline{x^2 + y^2 = 1}$

20/10/2011

1. Find the envelope of family of straight line $\frac{x}{a} + \frac{y}{b} = 1$ where a, b are parameters connected by the relation. $ab = c^2$.

Sol Given $ab = c^2$

$\Rightarrow b = c^2/a.$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 1 \rightarrow ①$

$\Rightarrow \cancel{\frac{x}{a} + \frac{y}{c^2}} = 1 \quad \frac{x}{a} + \frac{ya}{c^2} = 1$

$\Rightarrow \frac{c^2x + a^2y}{ac^2} = 1$

$\Rightarrow c^2x + a^2y = ac^2$

$\Rightarrow a^2y - ac^2 + xc^2 = 0.$

this is the Quadratic eqn

$b^2 - 4ac = 0$

$\Rightarrow (-c^2)^2 - 4yc^2x = 0$

$\Rightarrow c^4 - 4yc^2x = 0$

$\Rightarrow c^2 = 4yx.$

$\Rightarrow \underline{c^2 = 4xy}$

2. obtain the envelope of family of curves $\frac{x^2}{a^2}+\frac{y^2}{k^2-a^2}=1$ where a is parameter

Sol $\frac{x^2}{a^2}+\frac{y^2}{k^2-a^2}=1$

$$\Rightarrow \frac{x^2(k^2-a^2)+a^2y^2}{a^2(k^2-a^2)}=1$$

$$\Rightarrow x^2k^2-x^2a^2+a^2y^2=a^2k^2-a^4$$

$$\Rightarrow a^4+a^2(y^2-x^2-k^2)+x^2k^2=0.$$

This is Q. Eq. in terms of a^2

the Envelope is $b^2-4ac=0$.

$$\Rightarrow (y^2-x^2-k^2)^2-4x^2k^2=0$$

$$\Rightarrow [y^2-(x^2+k^2)]^2-4x^2k^2=0$$

$$[y^2-(x^2+k^2)]^2-(2xk)^2=0$$

$$\Rightarrow [y^2-(x^2+k^2)+2xk][y^2-(x^2+k^2)-2xk]=0$$

$$\Rightarrow [y^2-(x+k)^2][y^2-(x-k)^2]=0$$

$$(y+x+k)(y-(x+k))(y+(x-k))(y-(x-k))=0$$

3. find the envelope of circles: $x^2+y^2-2ax\cos\alpha-2ay\sin\alpha=c^2$ where α is parameter.

Sol $x^2+y^2-2ax\cos\alpha-2ay\sin\alpha=c^2$

$\rightarrow -2ax\cos\alpha-2ay\sin\alpha=c^2-(x^2+y^2)$

This is in the form of $A\cos\alpha+B\sin\alpha=C$

then envelope is $A^2+B^2=C^2$

$\Rightarrow (-2ax)^2+(-2ay)^2=[c^2-(x^2+y^2)]^2$

$\Rightarrow 4a^2(x^2+y^2)=[c^2-(x^2+y^2)]^2$

$\Rightarrow 2a\sqrt{(x^2+y^2)}=c^2-(x^2+y^2)$

$\therefore x^2+y^2+2a\sqrt{x^2+y^2}=c^2$

$\rightarrow (1)$

$=0 \rightarrow (2)$

(1) Pow.r.t α Q.E then

(2) $ax^2+bx+c=0$

$b^2-4ac=0$

(3) $A\cos\alpha+B\sin\alpha=C$

$A^2+B^2=C^2$