UNIT: 01 MATRIX THEORY

Definition of Rank, Normal form, Echelon form and Rank solution of system of linear equations.
Linearly dependent and Independent vectors.
Characteristic equations, characteristic roots / Eigen values and Eigen vectors.
Cayley Hamilton theorem and Applications.
Diagonalization
Reduction of Quadratic form to Canonical form.

Definition of Rank, Rank of Matrix, Echelon form and Normal form:

Matrix: A two dimensional arrangement of elements in m rows and n columns is called Matrix.

Rank of Matrix: A number "r" is said to be rank of a matrix if it possess the following two properties:

1. There exists atleast one minor order "r" whose determinant is non-zero.
2. All minor of order "r+1" if exists, all zeroes.

Q1. Find the rank of matrix A where $A = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Sol. $|A| = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$ $r = 3$ r = order

$\Rightarrow 3(12-15) - 4(6-12) + 5(5-8)$

$\Rightarrow 3(-3) - 4(-6) + 5(-3)$

$\Rightarrow -9 + 24 - 15$

$\Rightarrow -24 + 24$

$\Rightarrow 0$

$\therefore$ Rank of A is not 3.

SubMatrix $= \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$

$r = 2$

$\therefore$ Rank of A is 2.

Q2. Find Rank of matrix A where $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

Sol. $|A| = \begin{vmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{vmatrix}$ $r = 3$ r = order.

$2(72-72) - 3(48-48) + 4(36-36)$

$= 0$.

$\therefore$ Rank of A is not 3.

Take Submatrix $= \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$.

$\therefore$ Rank of A is not 2.

$\therefore$ Rank of A is 1 //.

Find the ~~rank of~~ value of λ such that rank of A is 2

where $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & \lambda & 5 \\ 4 & 8 & \lambda \end{bmatrix}$.

Since the Rank of A is 2.

$\boxed{|A|_{3\times 3} = 0}$

$\Rightarrow \begin{vmatrix} 1 & 2 & 4 \\ 2 & \lambda & 5 \\ 4 & 8 & \lambda \end{vmatrix} = 0$

$\Rightarrow 1(\lambda^2 - 40) - 2(2\lambda - 20) + 4(16 - 4\lambda) = 0$

$\Rightarrow \lambda^2 - 40 - 4\lambda + 40 + 64 - 16\lambda = 0$

$\Rightarrow \lambda^2 - 20\lambda + 64 = 0$

$$\lambda = \frac{20 \pm \sqrt{400 - 4(1)(64)}}{2(1)}$$

$$= \frac{20 \pm \sqrt{144}}{2}$$

$$\lambda = \frac{20 \pm 12}{2}$$

$$\lambda = \frac{32}{2}, \frac{8}{2}$$

$$\lambda = 16, 4$$

$\therefore \lambda = 16$ and 4 //

Q4. Find the value of 'b' of the matrix such that rank of A is 2 where $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{bmatrix}$.

Sol. Given; rank of A is 2.
Since rank of A is 2 $\Rightarrow \boxed{|A|_{3\times 3} = 0}$

$$\Rightarrow \begin{vmatrix} 1 & 5 & 4 \\ 0 & 3 & 2 \\ b & 13 & 10 \end{vmatrix} = 0$$

$$\Rightarrow 1(30-26) - 5(0-2b) + 4(0-3b) = 0$$

$$\Rightarrow 4 - 0 + 10b - 0 - 12b = 0$$

$$\Rightarrow 4 - 2b = 0$$

$$\Rightarrow \boxed{b = 2}.$$

Q5. Find the value of μ for which rank of A is '3' where

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

Sol. Since the rank of A is 3;

$$\Rightarrow \boxed{|A|_{4\times 4} = 0}$$

$$|A| = \begin{vmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{vmatrix}$$

$$\Rightarrow \mu \begin{vmatrix} \mu & -1 & 0 \\ 0 & \mu & -1 \\ 11 & -6 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 0 & -1 & 0 \\ 0 & \mu & -1 \\ -6 & -6 & 1 \end{vmatrix} + 0 \begin{vmatrix} 0 & \mu & 0 \\ 0 & 0 & -1 \\ -6 & 11 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & \mu \\ 0 & 0 \\ -6 & 11 \end{vmatrix}$$

$$\Rightarrow u\begin{vmatrix} u & -1 & 0 \\ 0 & u & -1 \\ 11 & -6 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 & 0 \\ 0 & u & -1 \\ -6 & -6 & 1 \end{vmatrix} + 0 - 0$$

$\Rightarrow u\left[u(u-6)-(-1)(0+11)-0(0-11u)\right] + 1\left[0(u+6)+1(0-6)+0\right]$

$\Rightarrow u\left[u^2-6u+11+1(-6)\right\} = 0$

$\Rightarrow u^3-6u^2+11u-6 = 0$

$\Rightarrow (u-1)(u-2)(u-3) = 0$

$\Rightarrow u-1=0 \ , \ u-2=0 \ , \ u-3=0$

$\Rightarrow u=1 \ , \ u=2 \ , \ u=3$

$\therefore$ Value of $u = 1, 2, 3$.

Find the value of k such that rank of A is '2' where

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

Given rank of A is 2.

$\Rightarrow \boxed{|A|_{3\times3} = 0}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{vmatrix} = 0$$

$\Rightarrow 1(10k-42)-2(20-21)+3(12-3k)=0$

$\Rightarrow 10k-42-2(-1)+3(12-3k)=0$

$10k-42+2+36-9k=0$

$\Rightarrow k-4=0$

$\Rightarrow \boxed{k=4}$

⇝ Rank of Matrix are of two types : ① Echelon form
② Normal form

① Echelon form : 1) Matrix A is said to be an echelon form if it satisfies the following conditions.

(i) Zero rows if any occurs then there should be below the non-zero rows.
(ii) First non-zero element in any non zero row is unity.
(iii) The the number of zeroes before the non-zero elements increase with the row numbers.
(iv) Every matrix can be transferred to Echelon form by applying Elementary row transformations.
(v) The Rank of a matrix in Echelon form is equal to the non-zero rows, it is denoted by "ρ".

Q1. Find the Rank of matrix where $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$ by using Echelon form.

Sol: Given $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$

By Echelon form $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 2 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 1

∴ Rank of A is 1 //.

Q2. Find Rank of Matrix $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$ by Echelon form.

Sol. Given $A = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$

$R_1 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 6 & 7 & 8 & 9 \\ 11 & 12 & 13 & 14 \\ 16 & 17 & 18 & 19 \end{bmatrix}$$

$R_2 \to R_2 - 6R_1$
$R_3 \to R_3 - 11R_1$
$R_4 \to R_4 - 16R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As number of non-zero rows = 2

$\therefore$ Rank of $A = 2$ //.

Q3. Find the rank of matrix A where $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$ by using Echelon form.

Sol: Given $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

$R_1 \leftrightarrow R_2 \sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 2 & -1 & 3 & 1 \\ 5 & 2 & 4 & 3 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 5R_1 \sim \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & -9 & 7 & -1 \\ 0 & -18 & 14 & -2 \end{bmatrix}$

$R_3 \to \dfrac{R_3}{-2} \sim \begin{bmatrix} 1 & 4 & 2 & 1 \\ 0 & -9 & 7 & -1 \\ 0 & 9 & -7 & 1 \end{bmatrix}$

$R_3 \to R_3 + R_2 = \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & -9 & 7 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ Rank of $A = 2$ as number of non-zero rows are 2.

Q4. Determine the value of k for which the matrix A is $\begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & k \end{bmatrix}$ by using Echelon form if rank is 3.

Sol. Given $A = \begin{bmatrix} 6 & 3 & 5 & 9 \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & K \end{bmatrix}$

$R_1 \to R_1 - R_2$ $\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 5 & 2 & 3 & 6 \\ 3 & 1 & 2 & K \end{bmatrix}$

$R_2 \to R_2 - 5R_1$
$R_3 \to R_3 - 3R_1$ $\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -3 & -7 & -9 \\ 0 & -2 & -4 & K-9 \end{bmatrix}$

$R_3 \to 2R_2 - 3R_3$ $\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -3 & -7 & -9 \\ 0 & 0 & -2 & -3k+9 \end{bmatrix}$

Since Rank of A is given as 3;

then $-3k+9=0$

$\Rightarrow 3K=9$

$\Rightarrow \boxed{K=3}$

∴ The value of k is '3' //.

Find the rank of matrix A where $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 1 \end{bmatrix}$.

Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 2 & 1 \end{bmatrix}$

$R_2 \rightarrow R_2 - 4R_1$

$R_3 \rightarrow R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_2$

$R_4 \rightarrow R_4 + \frac{2}{3}R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = 3$ as number of non-zero rows are 3 //.

Q6. Reduce the following matrix to Echelon form and find its rank

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

Sol: Given

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 \\ 2 & 3 & 4 & 5 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 \\ 0 & 3 & 0 & -1 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 6 & 8 \end{bmatrix}$$

$\therefore$ Rank of matrix is 3

Normal Form:

Applying Row and Column transformation It can be reduced into the form of $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$, $[I_r]$, $[I_r - 0]$.

where I_r is the unit matrix of order 'r'.

1. Find the rank of matrix A where $A = \begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ by using normal form.

Sol: Given $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$
$R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$C_2 \to C_3 - C_1$
$C_4 \to C_4 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & -3 & -1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 + 3C_2$
$C_4 \to C_4 + C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

As number of non-zero rows = 2.

$\therefore$ Rank of matrix $A = 2$. $\rho(A) = 2$.

Q2. $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ by normal form find rank.

Sol. Given $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

$C_1 \leftrightarrow C_2$

$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$

$R_3 \to R_3 - R_1$

$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$

$R_3 \to 2R_3 - R_2$

$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_2 \to \frac{C_2}{4} \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \to C_3 - 2C_1$

$C_4 \to C_4 + 2C_1$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$C_3 \to C_3 - 2C_2$

$C_4 \to C_4 - 6C_2$ $\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore \rho(A) = 2$

(2) Solution of System of Linear equations.

Consistent: System of equations is said to be consistent if it has solution otherwise inconsistent.

System of equations are of two types

1. Non-Homogeneous equations.
2. Homogeneous equations.

(1) Non-Homogeneous equations:

→ Let the Non-Homogeneous system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

→ Convert the equation in matrix equation form i.e

$$\boxed{AX = B}$$

→ Let $C = [A \,\vdots\, B]$ be the Augmented matrix.

→ Reduce the matrix into Echelon form

Rank of C $\rho(C)$ = Rank of A $\rho(A)$

Note: (1) If $\boxed{\rho(C) = \rho(A) = n}$ then the system is consistent. It has a unique solution.

(2) If $\boxed{\rho(C) = \rho(A) < n}$ then the system is consistent. It has infinite number of solutions.

(3) If $\boxed{\rho(C) \neq \rho(A)}$ then the system is inconsistent; It has no solution.

Q1. Discuss the consistency of the following equations.

$$x+2y+3z=6$$
$$5x-3y+2z=4$$
$$2x+4y-z=5$$
$$3x+2y+4z=9$$

Sol: The given system of equations $x+2y+3z=6$

$$5x-3y+2z=4$$
$$2x+4y-z=5$$
$$3x+2y+4z=9$$

⇒ Convert into Matrix equation form

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 5 & -3 & 2 \\ 2 & 4 & -1 \\ 3 & 2 & 4 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 6 \\ 4 \\ 5 \\ 9 \end{bmatrix}}$$

⇒ let $C = [A \vdots B]$ be the Augmented Matrix.

$$C = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 5 & -3 & 2 & 4 \\ 2 & 4 & -1 & 5 \\ 3 & 2 & 4 & 9 \end{bmatrix}$$

⇒ Reduce Matrix to Echelon form;

$R_2 \to R_2 - 5R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -13 & -13 & -26 \\ 0 & 0 & -7 & -7 \\ 0 & -4 & -5 & -9 \end{bmatrix}$$

$R_2 \to R_2/-13$

$R_3 \to R_3/-7$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & -4 & -5 & -9 \end{bmatrix}$$

$R_4 \to R_4 + 4R_2$ $\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$

$R_4 \to R_4 + R_3$

$\sim \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = C$

$\rho(C) = 3$, $\rho(A) = 3$, $n = 3$.

$\therefore \rho(C) = \rho(A) = n$.

Hence the system is consistent; It has a unique solution;

$x + 2y + 3z = 6$, $y + z = 2$, $z = 1$

$y + 1 = 2$

$y = 2 - 1$

$y = 1$

$x + 2(1) + 3(1) = 6$

$x + 2 + 3 = 6$

$x + 5 = 6$

$x = 6 - 5 = 1$

$x = 1$.

Substitute x, y, z in any one equation;

$x + 2y + 3z = 6$

$1 + 2(1) + 3(1) = 6$

$6 = 6$

$\therefore$ $x = 1$, $y = 1$ and $z = 1$ //.

Q2. Discuss the consistency of the following equations.

$$x - 3y - 8z = -10$$
$$3x + y - 4z = 0$$
$$2x + 5y + 6z = 13$$

Sol. Given system of equations; $x - 3y - 8z = -10$, $3x + y - 4z = 0$, $2x + 5y + 6z = 13$

$\Rightarrow$ Convert into Matrix form;

$$\underset{A}{\begin{bmatrix} 1 & -3 & -8 \\ 3 & 1 & -4 \\ 2 & 5 & 6 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} -10 \\ 0 \\ 13 \end{bmatrix}}$$

$\Rightarrow$ let $C = [A \,\vdots\, B]$ be the Augmented Matrix

$$C = \begin{bmatrix} 1 & -3 & -8 & -10 \\ 3 & 1 & -4 & 0 \\ 2 & 5 & 6 & 13 \end{bmatrix}$$

$\Rightarrow$ Reduce Matrix into Echelon form

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$
$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 10 & 20 & 30 \\ 0 & 11 & 22 & 33 \end{bmatrix}$$

$R_2 \to \frac{R_2}{10}$, $R_3 \to \frac{R_3}{11}$
$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\sim \begin{bmatrix} 1 & -3 & -8 & -10 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\rho(C) = 2$; $\rho(A) = 2$, $n = 3$

$\boxed{\rho(C) = \rho(A) < n}$ $\therefore$ System is consistent. It has infinite number of solutions.

$x - 3y - 8z = -10$

$y + 2z = 3$.

let $z = k$ (as there are 2 equations and 3 variables).

$y + 2z = 3$

$\Rightarrow y + 2k = 3$

$\Rightarrow y = 3 - 2k$

$x - 3y - 8z = -10$

$x = -10 + 3y + 8z$

$x = -10 + 3(3 - 2k) + 8k$

$x = -10 + 9 - 6k + 8k$

$x = 2k - 1,\ y = 3 - 2k,\ z = k$.

If we take $z = 1, 2, 3, \ldots$ i.e k.

$x = 2(1) - 1 = 1$

$= 2(2) - 1 = 3 \ldots$

$y = 3 - 2(1) = 1 \ldots$

$3 - 2(2) = -1 \ldots$

i.e we have infinite number of solutions!

3. $4x+9y+3z=6$
$2x+3y+z=2$
$2x+6y+2z=7$

Sol. The given system of linear equations;

$$4x+9y+3z=6$$
$$2x+3y+z=2$$
$$2x+6y+2z=7$$

⇒ Convert into Matrix form

$$\underbrace{\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}}_{B}$$

$$AX=B$$

⇒ let $C=[A \vdots B]$ be the Augmented Matrix

$$C=\begin{bmatrix} 4 & 9 & 3 & 6 \\ 2 & 3 & 1 & 2 \\ 2 & 6 & 2 & 7 \end{bmatrix}$$

⇒ Reduce Matrix into Echelon form

$R_2 \to 2R_2 - R_1$
$R_3 \to 2R_3 - R_1$

$$\sim \begin{bmatrix} 4 & 9 & 3 & 6 \\ 0 & -3 & -1 & -2 \\ 0 & 3 & 1 & 8 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 4 & 9 & 3 & 6 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$\rho(C)=3$; $\rho(A)=2$; $n=3$

$\rho(C) \neq \rho(A)$

∴ The system is inconsistent. It has no solution //

Q4. $5x+3y+7z=4$
$3x+26y+2z=9$
$7x+2y+10z=5$

Sol. Given system of linear equations;

$$5x+3y+7z=4$$
$$3x+26y+2z=9$$
$$7x+2y+10z=5$$

$\Rightarrow$ Convert into Matrix form

$$\underbrace{\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}}_{B}$$

$\Rightarrow$ Let $C=[A \vdots B]$ be the Augmented Matrix.

$$C=\begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}.$$

$\Rightarrow$ Reduce Matrix into Echelon form.

$R_1 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 7 & 2 & 10 & 5 \\ 3 & 26 & 2 & 9 \\ 5 & 3 & 7 & 4 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} 2 & -1 & 3 & 1 \\ 3 & 26 & 2 & 9 \\ 5 & 3 & 7 & 4 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} 1 & 5 & 1 & 2 \\ 0 & 22 & -2 & 6 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 5 & 1 & 2 \\ 0 & -11 & 1 & -3 \\ 0 & 22 & -2 & 6 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 5 & 1 & 2 \\ 0 & -11 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2,\ \rho(C) = 2,\ n = 3$

$\therefore$ System is consistent as $\rho(C) = \rho(A) < n$. It has infinite number of solutions.

$$x + 5y + z = 2$$
$$0x - 11y + z = -3$$

let $z = k$

$-11y + z = -3$

$z - 11y = -3$

$y = \dfrac{-3-k}{-11}$

$1x + 5y + 1z = 2$

$1x + 5\left(\dfrac{-3-k}{-11}\right) + z = 2$

$x - \dfrac{15 - 5k}{11} + z = 2$

$11x - 15 - 5k + 11z = 22$

$11x = 22 + 15 - 5k + 11z$

$11x = 37 - 5k + 11z$

$x = 37 - 5k + 11k$

$x = 37 + 6k$

If we take $z = 1, 2, 3$ i.e

$x = 37 + 6 = 43 \ldots$

$y = \dfrac{-3-1}{-11} = \dfrac{-4}{-11} = \dfrac{4}{11}$

$\therefore$ we have infinite no of solutions.

Q5. Test for consistency for the following pair of equations

$$4x+9y+3z=6$$
$$2x+3y+z=2$$
$$2x+6y+2z=7$$

Sol. The Given system of equations are $4x+9y+3z=6$
$2x+3y+z=2$
$2x+6y+2z=7$

⇒ Convert into Matrix equation form;

i.e
$$\underset{A}{\begin{bmatrix} 4 & 9 & 3 \\ 2 & 3 & 1 \\ 2 & 6 & 2 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 6 \\ 2 \\ 7 \end{bmatrix}}$$

⇒ Let $C=[A \vdots B]$ be the Augmented Matrix;

$$C = \begin{bmatrix} 4 & 9 & 3 & 6 \\ 2 & 3 & 1 & 2 \\ 2 & 6 & 2 & 7 \end{bmatrix}$$

⇒ Reduce the Matrix into echelon form;

$$\begin{matrix} R_2 \to 2R_2 - R_1 \\ R_3 \to 2R_3 - R_1 \end{matrix} \sim \begin{bmatrix} 4 & 9 & 3 & 6 \\ 0 & -3 & -1 & -2 \\ 0 & 3 & 1 & 8 \end{bmatrix}$$

$$R_3 \to R_3 + R_2 \sim \begin{bmatrix} 4 & 9 & 3 & 6 \\ 0 & -3 & -1 & -2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

$\rho(A)=2$, $\rho(C)=3$, $n=3$

$\therefore \rho(A) \neq \rho(C)$

$\therefore$ The system is Inconsistent. It has no solution.

Q6. For what values of a and b do the system of linear equations
$$x+y+z=6,$$
$$x+2y+3z=10$$
$$x+2y+az=b$$
1) has unique solution.
2) has infinite solution.
3) has no solution.

Sol: Given equations
$$x+y+z=6$$
$$x+2y+3z=10$$
$$x+2y+az=b.$$

$\Rightarrow$ Convert into Matrix form;

i.e
$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & a \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 6 \\ 10 \\ b \end{bmatrix}}$$

$\Rightarrow$ let $C=[A \vdots B]$ be the Augmented matrix;
$$C=\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 1 & a & b \end{array}\right]$$

$\Rightarrow$ Reduce the Matrix into Echelon form;

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & a-1 & b-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-1-2 & b-6-4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & a-3 & b-10 \end{bmatrix}$$

(1) Unique solution; $\boxed{\rho(C) = \rho(A) = n}$

then $n=3 \Rightarrow \rho(A) = 3, \rho(C) = 3$

$a-3 \neq 0$, $b-10 = 0$

$a \neq 3$, ~~$b \neq 10$~~ $b = 10$.

(2) Infinite solution ; $\boxed{\rho(C) = \rho(A) < n}$

then $a-3=0$, $b-10=0$

$a=3$, $b=10$

(3) No solution ; $\boxed{\rho(C) \neq \rho(A)}$

$a-3=0$, $b-10 \neq 0$

$a=3$, $b \neq 10$.

$\therefore \rho(C) = 3$, $\rho(A) = 2$

Linear Dependent and Independent of Vectors.

Note: ① If $|A|=0$ then the given vectors are linearly dependent.
② If $|A| \neq 0$ then the given vectors are linearly Independent.

Q1. Check the following vectors are linearly dependent or Independent.
$(1,0,1), (1,1,2), (2,2,4)$.

Sol: Given vectors $x_1=[1,0,1]$, $x_2=[1,1,2]$, $x_3=[2,2,4]$

$$A=\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 1(4-4)-1(0-2)+2(0-1)$$
$$= 1(0)-1(-2)+2(-1)$$
$$= 0+2-2$$
$$= 0$$

$\boxed{|A|=0}$

$\therefore$ The given vectors are linearly dependent.

Q2. Check if the following vectors are linearly dependent or Independent.
$[2,2,0], [3,0,2], [2,-2,2]$.

Sol: Given vectors $x_1=[2,2,0]$, $x_2=[3,0,2]$, $x_3=[2,-2,2]$

$$A=\begin{bmatrix} 2 & 3 & 2 \\ 2 & 0 & -2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$|A|=\begin{vmatrix} 2 & 3 & 2 \\ 2 & 0 & -2 \\ 0 & 2 & 2 \end{vmatrix} \quad 2(0+4)-3(4-0)+2(4-0)$$
$$\Rightarrow 8-12+8$$
$$\Rightarrow 16-12$$
$$\Rightarrow 4 \neq 0$$

$\therefore \boxed{|A| \neq 0}$

$\therefore$ The given vectors are linearly Independent.

(4) Characteristic Equation ; Characteristic roots ; Eigen values and Eigen vectors.

→ **Characteristic equation:** If "A" is a square matrix of order "n". It can form the matrix $[A-\lambda I]$.
$|A-\lambda I| = 0$ is called characteristic equation on expansion;
$(-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \dots + k_n = 0$
is called characteristic equation.
The roots of this equation is called 'Characteristic roots' or 'Eigen values'. and λ is scalar and I is unit matrix of order 'n'.

→ characteristic roots or Eigen values are of three types:

1. The roots are distinct (unequal)
2. Any two roots are equal.
3. Three roots are equal.

Note: (1) The sum of Eigen values = The Sum of principal Diagonal Elements.

Note (2): The Product of the Eigen values = $|A|$ [Determinant of A].

Q1. Find Sum and product of Eigen values of |A| where

$$A = \begin{bmatrix} 10 & 0 & 8 \\ 4 & 9 & 6 \\ 2 & 7 & 5 \end{bmatrix}$$

Sol. Given $A = \begin{bmatrix} 10 & 0 & 8 \\ 4 & 9 & 6 \\ 2 & 7 & 5 \end{bmatrix}$.

The Sum of Eigen values = Sum of principal diagonal elements

$= 10 + 9 + 5$

$\boxed{Sum = 24}$

The product of eigen values $= \begin{vmatrix} 10 & 0 & 8 \\ 4 & 9 & 6 \\ 2 & 7 & 5 \end{vmatrix}$

$\Rightarrow 10(45-42) - 0(20-12) + 8(28-18)$

$\Rightarrow 10(3) - 0 + 8(10)$

$\Rightarrow 30 + 80 = 110$

$\boxed{|A| = 110}$ //.

Q2. Find the Sum of Eigen values where $A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 5 & 6 & 3 & 2 \\ 7 & 6 & 1 & 3 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

Sol. Given $A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 5 & 6 & 3 & 2 \\ 7 & 6 & 1 & 3 \\ 1 & 0 & 0 & 2 \end{bmatrix}$

Sum $= 2 + 6 + 1 + 2$

$= 11$

//.

Q3. If the Sum of Eigen values of Matrix $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & k & 2 \\ -1 & 2 & 2k \end{bmatrix}$ is 10 find value of k.

Sol. Given $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & k & 2 \\ -1 & 2 & 2k \end{bmatrix}$

$1 + k + 2k = 10$

$3k + 1 = 10$

$3k = 9$

$\boxed{k = 3}$

Q4. Find the characteristic equation; characteristic roots; Eigen values and Eigen vectors of the Matrix A where

$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Sol. Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

let λ be the characteristic root of the eigen value.

The characteristic equation $|A - \lambda I| = 0$

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (1-\lambda)[(5-\lambda)(1-\lambda)-1] - 1[1-\lambda-3] + 3[1-3(5-\lambda)] = 0$

$\Rightarrow (1-\lambda)[5-5\lambda-\lambda+\lambda^2-1] - 1[1-\lambda-3] + 3[1-15+3\lambda] = 0$

$\Rightarrow (1-\lambda)[4-6\lambda+\lambda^2] - 1[-\lambda-2] + 3[-14+3\lambda] = 0$

$\Rightarrow \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 2 + 9\lambda - 42 = 0$

$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0 \Rightarrow$

$\Rightarrow$

$\lambda^3 - 7\lambda^2 - 36 = 0$.

$\therefore \lambda = -2, 3, 6$

Characteristic roots are $-2, 3, 6$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the Eigen vector corresponding to Eigen values.

According to Eigen vectors.

$AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

Case (1): If $\lambda = -2$ then eq^n (1) becomes.

$$\Rightarrow \begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 3x_1 + x_2 + 3x_3 &= 0 \\ 1x_1 + 7x_2 + 1x_3 &= 0 \\ 3x_1 + x_2 + 3x_3 &= 0 \end{aligned}\right\}$$

$$\begin{matrix} 7 & 1 & 1 & 7 \\ 1 & 3 & 3 & 1 \end{matrix}$$

$$\frac{x_1}{21-1} = \frac{x_2}{3-3} = \frac{x_3}{1-21}$$

$$\frac{x_1}{2\cancel{0}} = \frac{x_2}{0} = \frac{x_3}{-2\cancel{0}}$$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Case ②: If $\lambda=3$; Sub in ①.

$$\Rightarrow \begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} -2x_1+1x_2+3x_3&=0 \\ 1x_1+2x_2+1x_3&=0 \\ 3x_1+1x_2-2x_3&=0 \end{aligned}\right\}$$

$$\begin{matrix} 2 & 1 & 1 & 2 \\ 1 & -2 & 3 & 1 \end{matrix}$$

$$\frac{x_1}{-4-1} = \frac{x_2}{3+2} = \frac{x_3}{1-6}$$

$$\frac{x_1}{\cancel{-5}} = \frac{x_2}{\cancel{5}} = \frac{x_3}{\cancel{-5}}$$

$$x_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Case ③: If $\lambda=6$; Sub in ①

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-5x_1+x_2+3x_3=0$$
$$1x_1-1x_2+1x_3=0$$
$$3x_1+1x_2-5x_3=0$$

$$\begin{matrix} -1 & 1 & 1 & -1 \\ 1 & -5 & 3 & 1 \end{matrix}$$

$$\frac{x_1}{5-1} = \frac{x_2}{3+5} = \frac{x_3}{1+3}$$

$$\frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4} \qquad x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

∴ The characteristic roots $= \lambda = 2, 3, 6$ and eigen values

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

//.

Q5. Find Eigen values, Eigen vectors of the following matrix A :
where $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol. Given matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

let λ be the characteristic root or Eigen value then

$|A-\lambda I|=0 \Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow 8-\lambda\left[(7-\lambda)(3-\lambda)-16\right] + 6\left[-6(3-\lambda)+8\right] + 2\left[24-2(7-\lambda)\right] = 0$

$\Rightarrow 8-\lambda\left[5-10\lambda+\lambda^2\right] + 6\left[6\lambda-10\right] + 2\left[2\lambda+10\right] = 0$

$\Rightarrow 8-\lambda\left[\lambda^2-10\lambda+5\right] + 6\left[6\lambda-10\right] + 2\left[2\lambda+10\right] = 0$

$\Rightarrow 8\lambda^2-80\lambda+40-\lambda^3+10\lambda^2-5\lambda+36\lambda-60+4\lambda+20 = 0$

$\Rightarrow -\lambda^3+18\lambda^2-45\lambda=0$

$\Rightarrow \lambda^3-18\lambda^2+45\lambda=0$

$\lambda = 0, 3, 15$

$\therefore$ The eigen values are $\lambda = 0, 3, 15$.

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to Eigen values

According to eigen vector $AX = X\lambda$

$AX - X\lambda = 0$

$(A-\lambda I)X = 0$.

$$\begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—(1)}.$$

Case (1): If $\lambda = 0$ Sub in (1).

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left.\begin{aligned} 8x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 7x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 3x_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{matrix} 7 & -4 & -6 & 7 \\ -4 & 3 & 2 & -4 \end{matrix}$$

$$\Rightarrow \frac{x_1}{21-16} = \frac{x_2}{-8+18} = \frac{x_3}{24-14}$$

$$\Rightarrow \frac{x_1}{5} = \frac{x_2}{10} = \frac{x_3}{10}$$

$$\Rightarrow \frac{x_1}{1} = \frac{x_2}{2} = \frac{x_2}{2} \qquad \therefore \; x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Case (2): If $\lambda = 3$ Sub in (1).

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\left.\begin{aligned} 5x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 + 4x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 + 0x_3 &= 0 \end{aligned}\right\}$$

$$4 \quad -4 \quad -6 \quad 4$$
$$-4 \quad 0 \quad 2 \quad -4$$

$$\frac{x_1}{0-16} = \frac{x_2}{-8-0} = \frac{x_3}{24-8}$$

$$\frac{x_1}{-16} = \frac{x_2}{-8} = \frac{x_3}{16}$$

$$x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case 3: If $\lambda = 15$; sub in 1.

$$\begin{bmatrix} 7 & -6 & 2 \\ -6 & -8 & -14 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -7x_1 - 6x_2 + 2x_3 &= 0 \\ -6x_1 - 8x_2 - 4x_3 &= 0 \\ 2x_1 - 4x_2 - 12x_3 &= 0 \end{aligned}\right\} \quad \frac{x_1}{96-16} = \frac{x_2}{-8-12} = \frac{x_3}{24+16}$$

$$-8 \quad -4 \quad -6 \quad -8$$
$$-4 \quad -12 \quad 2 \quad -4$$

$$\frac{x_1}{80} = \frac{x_2}{-80} = \frac{x_3}{40}$$

$$\therefore x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore$ Eigen values are $\lambda = 0, 3, 15$.

Eigen vectors $x_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

Q6. Find the Eigen values and Eigen vectors of the following matrix A where $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Sol. Given matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$

let λ be the characteristic root or eigen value then

$|A-\lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{vmatrix} = 0$$

$\Rightarrow -2-\lambda[(1-\lambda)(0-\lambda)-12]-2[2(0-\lambda)-6)-3(-4+1(1-\lambda))=0$

$\Rightarrow -2-\lambda(-\lambda+\lambda^2-12)-2(-2\lambda-6)-3(-3-\lambda)=0$

$\Rightarrow -2-\lambda(\lambda^2-\lambda-12)-2(-2\lambda-6)-3(-\lambda-3)=0$

$\Rightarrow -2\lambda^2+2\lambda+24-\lambda^3+\lambda^2+12\lambda+4\lambda+12+3\lambda+9=0$

$\Rightarrow -\lambda^3-\lambda^2+21\lambda+45=0$

$\Rightarrow \lambda^3+\lambda^2-21\lambda-45=0$ is a characteristic equation.

characteristic roots or eigen values are

$\lambda = 5, -3, -3$.

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ Be the Eigen vectors corresponding to Eigen values λ.

$AX = X\lambda$

$AX - \lambda X = 0$

$(A-\lambda I)X = 0$.

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

Case (1): If $\lambda = 5$; Sub in (1).

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -7x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 - 4x_2 - 6x_3 &= 0 \\ -1x_1 - 2x_2 - 5x_3 &= 0 \end{aligned}\right\}$$

$$\begin{matrix} -4 & -6 & 2 & -4 \\ -2 & -5 & -1 & -2 \end{matrix} \Rightarrow \frac{x_1}{20-12} = \frac{x_2}{6+10} = \frac{x_3}{-4-4}$$

$$\Rightarrow \frac{x_1}{8} = \frac{x_2}{16} = \frac{x_3}{-8}$$

$$\therefore x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Case (2): If $\lambda = -3$; Sub in (1)

$$\begin{bmatrix} \cancel{-2+}1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 1x_1 + 2x_2 - 3x_3 &= 0 \\ 2x_1 + 4x_2 - 6x_3 &= 0 \\ -1x_1 - 2x_2 + 3x_3 &= 0 \end{aligned}\right\}$$

$$\begin{matrix} 4 & -6 & 2 & 4 \\ -2 & 3 & -1 & -2 \end{matrix} \Rightarrow \frac{x_1}{12-12} = \frac{x_2}{6-6} = \frac{x_3}{-4+4} \Rightarrow \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\Rightarrow x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

So let $x_3 = k_1$, $x_2 = k_2$,

$\Rightarrow 1x_1 + 2x_2 - 3x_3 = 0$

$x_1 = 3x_3 - 2x_2$

$x_1 = 3k_1 - 2k_2$, $x_2 = k_2$, $x_3 = k_1$

Linear combination;

$k_1 x_1 + k_2 x_2 + k_3 x_3 + \cdots = 0$

$k_1 [3, 0, 1] + k_2 [-2, 1, 0] = [0, 0, 0]$

$x_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

$\therefore$ Eigen value $\lambda = 5, -3, -3$

and eigen vectors $x_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $x_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

(5) Cayley Hamilton Theorem and Applications

Statement: Every square matrix satisfies its characteristic equation or If 'A' is a square matrix of order 'n' and if $|A-\lambda I| = (-1)^n \lambda^n + k_1 \lambda^{n-1} + k_2 \lambda^{n-2} + \cdots$

$+ \cdots k_{n-1}\lambda + k_n = 0$ such that

$$k_0 A^n + k_1 A^{n-1} + \cdots + k_{n-1} A + k_n I = 0 \quad \text{--- (1)}$$

To find A^{-1}: Multiply by A^{-1} of equation (1).

$$k_0 A^n A^{-1} + k_1 A^{n-1} A^{-1} + \cdots + k_{n-1} A A^{-1} + k_n A^{-1} I = 0$$

$$k_0 A^{n-1} + k_1 A^{n-2} + \cdots k_{n-1} I = -k_n A^{-1}$$

$$\therefore A^{-1} = \frac{1}{k_n}\left[k_0 A^{n-1} + k_1 A^{n-2} + k_2 A^{n-3} + \cdots k_{n-1} I\right)$$

To find the power of A: Multiply by A^{k-n} of equation (1).

$$k_0 A^n A^{k-n} + k_1 A^{n-1} A^{k-n} + \cdots k_{n-1} A A^{k-n} = 0$$

$$k_0 A^k + k_1 A^{k-1} + \cdots + k_{n-1} A^{k-n-1} + k_n A^{k-n} = 0$$

$$A^k = \frac{1}{k_0}\left[k_1 A^{k-1} + \cdots + k_{n-1} A^{k-n+0} + k_n A^{k-n}\right]$$

$$\begin{pmatrix} AA^{-1} = I \\ A^{-1} I = A^{-1} \end{pmatrix}$$

Q1. Verify Cayley Hamilton Theorem for the matrix A and also find A^{-1} and A^4 where $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$

let `λ' be the characteristic root or Eigen value.

The characteristic equation $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -1 \\ 3 & 1-\lambda & 0 \\ -2 & 1 & 4-\lambda \end{vmatrix} = 0$$

$(1-\lambda)\left[(1-\lambda)(4-\lambda)-0)\right] - 2\left[3(4-\lambda)-0\right] + (-1)\left[3+2(1-\lambda)\right] = 0$

$1-\lambda\left[4-\lambda-4\lambda+\lambda^2\right] - 2\left[12-3\lambda\right] - 1\left[3+2-2\lambda\right] = 0$

$1-\lambda\left[\lambda^2-5\lambda+4\right] - 2\left[-3\lambda+12\right] - 1\left[5-2\lambda\right] = 0$

$\lambda^2-5\lambda+4-\lambda^3+5\lambda^2-4\lambda+6\lambda-24-5+2\lambda = 0$

$-\lambda^3+6\lambda^2-\lambda-25 = 0$

$\Rightarrow \lambda^3-6\lambda^2+\lambda+25 = 0$

This is the characteristic equation's ~~eigen values~~.

If Cayley Hamilton Theorem is satisfied the above equation is $A^3-6A^2+A+25I = 0 \longrightarrow (1)$

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix}$$

$$A^3 = A^2 \times A = \begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 16 & -29 \\ 33 & 16 & -18 \\ -40 & 5 & 79 \end{bmatrix}.$$

$$A^3 - 6A^2 + A + 25I = 0$$

$$\begin{bmatrix} 28 & 16 & -29 \\ 33 & 16 & -18 \\ -40 & 5 & 79 \end{bmatrix} - 6\begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + 25\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 28 & 16 & -29 \\ 33 & 16 & -18 \\ -40 & 5 & 79 \end{bmatrix} + \begin{bmatrix} -54 & -18 & +30 \\ -36 & -42 & 18 \\ 42 & -6 & 108 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$\therefore A^3 - 6A^2 + A + 25I = 0$

$\therefore$ A satisfies characteristic equation.

$\therefore$ A satisfies Cayley Hamilton Theorem.

→ To find A^{-1}: Multiply by A^{-1} of ①

$$A^{-1}(A^3 - 6A^2 + A + 25I) = A^{-1} \times 0$$

$$A^3A^{-1} - 6A^2A^{-1} + AA^{-1} + 25A^{-1}I = 0$$

$$A^2AA^{-1} - 6A \cdot AA^{-1} + I + 25A^{-1} = 0$$

$$A^2I - 6AI + I + 25A^{-1} = 0$$

$$25A^{-1} = -A^2I + 6AI - I = -A^2 + 6A - I$$

$$A^{-1} = \frac{-1}{25}(A^2 - 6A + I)$$

$$A^{-1} = \frac{-1}{25}\left\{\begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix} - 6\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{-1}{25}\begin{bmatrix} 4 & -9 & 11 \\ -12 & 2 & -3 \\ 5 & -5 & -5 \end{bmatrix}.$$

$\rightarrow$ To find A^4:

from ① $A^3 - 6A^2 - A - 25I = 0$

Multiply by A.

$A^4 - 6A^3 - A^2 - 25AI = 0$

$A^4 = 6A^3 - A^2 - 25A$

$$A^4 = 6\begin{bmatrix} 28 & 16 & -29 \\ 33 & 16 & -18 \\ -40 & 5 & 79 \end{bmatrix} - \begin{bmatrix} 9 & 3 & -5 \\ 6 & 7 & -3 \\ -7 & 1 & 18 \end{bmatrix} - 25\begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & 0 \\ -2 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 134 & 43 & -144 \\ 117 & 64 & -105 \\ -183 & 4 & 356 \end{bmatrix}.$$

Q2. Find A^{-1} by using Cayley Hamilton theorem where

$$A = \begin{bmatrix} -1 & +1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

let λ be the characteristic root or eigen value equation

The characteristic equation is $|A - \lambda I| = 0$.

~~If Cayley Hamilton theorem is satisfied the above~~

The characteristic equation ; $|A-\lambda I|=0$.

$$\begin{vmatrix} -1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow -1-\lambda\left[(1-\lambda)(2-\lambda)-1\right]-1\left[0-2\right]+0\left[0-2(1-\lambda)\right]=0$

$\Rightarrow -1-\lambda\left[2-\lambda-2\lambda+\lambda^2-1\right]-1\left[-2\right]+0=0$

$\Rightarrow -1-\lambda\left[2-3\lambda+\lambda^2-1\right]+2=0$

$\Rightarrow -1-\lambda\left[\lambda^2-3\lambda+1\right]+2=0$

$\Rightarrow -\lambda^2+3\lambda-1-\lambda^3+3\lambda^2-\lambda+2=0$

$\Rightarrow -\lambda^3+2\lambda^2+2\lambda+1=0$

$\Rightarrow \lambda^3-2\lambda^2-2\lambda-1=0$

This is the characteristic equation or eigen value equation.

If Cayley Hamilton theorem is satisfied the above equation

is $A^3-2A^2-2A-I=0 \longrightarrow (1)$

To find A^{-1}: Multiply by A^{-1} of (1);

$A^{-1}(A^3-2A^2-2A-I)=A^{-1}\times 0$

$A^{-1}A^3-2A^2A^{-1}-2AA^{-1}-IA^{-1}=0$

$A^{-1}AA^2-2AAA^{-1}-2AA^{-1}-IA^{-1}=0$ $\quad AA^{-1}=I$

$A^2I-2AI-2I+IA^{-1}=0$

$A^2I-2AI-2I+A^{-1}=0 \Rightarrow A^2-2A-2I+A^{-1}$

$A^2I-2AI-2I=-A^{-1}$ $\quad A^{-1}=-A^2+2A+2I$

$$A^{-1}=\frac{A^2I-2AI-2I}{(-1)}$$

$A^{-1}=-A^2I+2AI+2I$.

$$A^2 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \\ 2 & 5 & 5 \end{bmatrix}.$$

$$-A^2 = \begin{bmatrix} -1 & 0 & -1 \\ -2 & -2 & -3 \\ -2 & -5 & -5 \end{bmatrix}, \quad 2AI = 2\begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 2 & 0 \\ 0 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

$$\therefore A^{-1} = -A^2 I + 2AI + 2I$$

$$= \begin{bmatrix} -1 & 0 & -1 \\ -2 & -2 & -3 \\ -2 & -5 & -5 \end{bmatrix} + \begin{bmatrix} -2 & 2 & 0 \\ 0 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

$$= \begin{bmatrix} -1 & 2 & -1 \\ -2 & 2 & -1 \\ 2 & -3 & 1 \end{bmatrix} //$$

Q3. Using Cayley Hamilton theorem, find A^4 where A equals

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}$$

1. Given $A = \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}$

let λ be the characteristic root or Eigen value.

The characteristic equation $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 0 & 3 \\ 8 & 1-\lambda & -7 \\ -3 & 0 & 8-\lambda \end{vmatrix} = 0$$

$\Rightarrow (-1-\lambda)\left[(1-\lambda)(8-\lambda)+0\right]-0\left[8(8-\lambda)-21\right]+3\left[0+3(1-\lambda)\right]=0$

$\Rightarrow (-1-\lambda)\left[8-\lambda-8\lambda+\lambda^2\right]-0+3\left[3-3\lambda\right]=0$

$\Rightarrow -\lambda^2+9\lambda-8-\lambda^3+9\lambda^2-8\lambda+9-9\lambda=0$

$\Rightarrow -\lambda^3+8\lambda^2-8\lambda+1=0$

$\Rightarrow \lambda^3-8\lambda^2+8\lambda-1=0$ is the characteristic equation.

If Cayley Hamilton theorem is satisfied the above equation is $A^3-8A^2+8A-I=0 \longrightarrow (1)$.

To find A^4:- from (1) $A^3-8A^2+8A-I=0$

Multiply by A

$A(A^3-8A^2+8A-I)=A\times 0$

$A^4-8A^3+8A^2-AI=0$

$A^4=8A^3-8A^2+AI$

$A^4=8A^3-8A^2+A$

$A^3=A^2\times A$

$$A^2=A\times A=\begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}\begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}=\begin{bmatrix} -8 & 0 & 21 \\ 21 & 1 & -39 \\ -21 & 0 & 55 \end{bmatrix}$$

$A^2 \times A$

$$A^3 = \begin{bmatrix} -8 & 0 & 21 \\ 21 & 1 & -39 \\ -21 & 0 & 55 \end{bmatrix} \times \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 55 & 0 & 144 \\ 104 & 1 & -256 \\ -144 & 0 & 337 \end{bmatrix}$$

$$8A^3 = \begin{bmatrix} 440 & 0 & 1152 \\ 832 & 8 & -2048 \\ -1152 & 0 & 2696 \end{bmatrix}$$

$$8A^2 = \begin{bmatrix} -64 & 0 & 168 \\ 168 & 8 & 312 \\ -168 & 0 & 440 \end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix} 440 & 0 & 1152 \\ -832 & 8 & -2048 \\ -1152 & 0 & 2696 \end{bmatrix} - \begin{bmatrix} -64 & 0 & 168 \\ 168 & 8 & 312 \\ -168 & 0 & 440 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 3 \\ 8 & 1 & -7 \\ -3 & 0 & 8 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 503 & 0 & 987 \\ -992 & 1 & -2367 \\ -987 & 0 & 2264 \end{bmatrix} //$$

Q4. Find characteristic equation of matrix A where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by
$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol. Given $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

let λ be the characteristic root of equation then characteristic equation $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0.$$

$\Rightarrow 2-\lambda[(1-\lambda)(2-\lambda) - 0] - 1[0-0] + 1[0 - 1(1-\lambda)] = 0$

$\Rightarrow 2-\lambda[\lambda^2 - 3\lambda + 2] - 0 + 1[\lambda - 1] = 0$

$\Rightarrow 2\lambda^2 - 6\lambda + 4 - \lambda^3 + 3\lambda^2 - 2\lambda + \lambda - 1 = 0$

$\Rightarrow -\lambda^3 + 5\lambda^2 - 8\lambda + \lambda + 3 = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ is the characteristic equation.

If Cayley Hamilton theorem is satisfied then 'A' satisfy the above equation: $A^3 - 5A^2 + 7A - 3I = 0$

$\Rightarrow A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

$\Rightarrow A^5[A^3 - 5A^2 + 7A - 3I] + A^4 - 5A^3 + 7A^2 - 3A + A^2 + A + I$

$\Rightarrow A^5[A^3 - 5A^2 + 7A - 3I] + A[A^3 - 5A^2 + 7A - 3I] + A^2 + A + I$

$\Rightarrow A^5[0] + A[0] + A^2 + A + I$

$\Rightarrow A^2 + A + I$.

$$A^2 = A \times A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow A^2 + A + I$.

$$\Rightarrow \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix} //$$

Q5. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ and also find A^{-1}.

Sol. Given $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}_{2\times 2}$

let λ be the characteristic root or Eigen value.

the characteristic eq^n $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 5-\lambda & 4 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$$5 - \lambda(2-\lambda) - 4 = 0$$

$$10 - 5\lambda - 2\lambda + \lambda^2 - 4 = 0$$

$\lambda^2 - 7\lambda + 6 = 0$ is the characteristic equation

If Cayley Hamilton theorem is satisfied then A becomes

$$A^2 - 7A + 6I = 0$$

$$A^2 = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix}$$

$$A^2 - 7A + 6I = 0 \longrightarrow (1)$$

$$\begin{bmatrix} 29 & 28 \\ 7 & 8 \end{bmatrix} - \begin{bmatrix} 35 & 28 \\ 7 & 14 \end{bmatrix} + \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore A^2 - 7A + 6I = 0$$

$\therefore$ Cayley Hamilton theorem is verified.

To find A^{-1}: Multiply by A^{-1} of (1).

$$A^{-1}(A^2 - 7A + 6I) = A^{-1} \times 0$$

$$A^{-1}A^2 - 7AA^{-1} + 6A^{-1}I = 0$$

$$A^{-1}A \cdot A - 7AA^{-1} + 6A^{-1}I = 0$$

$$A - 7I + 6A^{-1} = 0$$

$$6A^{-1} = -A + 7I$$

$$A^{-1} = -\frac{1}{6}[A - 7I]$$

$$A^{-1} = \frac{-1}{6}\left\{ \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \right\}$$

$$= \frac{-1}{6}\begin{Bmatrix} -2 & 4 \\ 1 & -5 \end{Bmatrix}$$

$$= \begin{Bmatrix} +1/3 & -2/3 \\ -1/6 & 5/6 \end{Bmatrix}$$

⑥ Diagonalisation

Orthogonal Matrix: A Matrix A is said to be orthogonal matrix if $AA^T = I$

We know that $AA^{-1} = A^{-1}A = I$

$$\Rightarrow A^{-1}A = AA^T$$

$$\Rightarrow A^{-1} = A^T$$

They are of two types:

1. Non-symmetric Matrix.
2. Symmetric Matrix.

① **Non Symmetric Matrix:** let 'A' be a square matrix (non-symmetric) of order 'n'. 'A' is said to be diagonizable if there exists a non-singular square matrix 'B' such that "$B^{-1}AB$" is known as diagonal matrix.

$B^{-1}AB$ is also known as 'Diagonalisation of A'. It is denoted by 'D' i.e $\boxed{D = B^{-1}AB}$.

Working Rule: ① To find Eigen values.

② To find Eigen vectors.

③ To find Modal matrix 'B' such that $B = [X_1\ X_2\ X_3]$

④ To find B^{-1} i.e $B^{-1} = \dfrac{\text{adj } B}{|B|}$.

⑤ To find diagonal matrix $D = B^{-1}AB$.

② **Symmetric Matrix:** let 'A' be a real symmetric matrix of order 'n'; apply the similarity transformation $B^{-1}AB = D = B^TAB$ is known as orthogonal transformation.

Working Rule: ① To find Eigen values.

② To find Eigen vectors.

③ To check if the vectors are orthogonal or not i.e

$X_1X_2 = X_2X_3 = X_3X_1 = 0$

④ To find modal matrix i.e $[x_1 \; x_2 \; x_3]$.

⑤ To find Normalizing matrix $B = \left[\frac{x_1}{\|x_1\|}, \frac{x_2}{\|x_2\|}, \frac{x_3}{\|x_3\|}\right]$

where $\|x\| = \sqrt{a^2+b^2+c^2}$

⑥ To find B^T.

⑦ To find Diagonal matrix $D = B^TAB$.

Q1. Diagonalize the matrix A where $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

Sol. Given $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

let 'λ' be the characteristic root or Eigen value then characteristic equation $|A-\lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow 1-\lambda[(2-\lambda)(1-\lambda)-1] - 1[-1(1-\lambda)-0] - 2[-1-0] = 0$

$\Rightarrow 1-\lambda[2-2\lambda+\lambda+\lambda^2-1] - 1[1+\lambda] + 2 = 0$

$\Rightarrow 1-\lambda[\lambda^2-3-\lambda] - 1-\lambda+2 = 0$

$\Rightarrow \lambda^2-3-\lambda-\lambda^3+3\lambda+\lambda^2-1-\lambda+2 = 0$

$\Rightarrow -\lambda^3+2\lambda^2+\lambda-2 = 0$

$\Rightarrow \lambda^3-2\lambda^2-\lambda+2 = 0$ is the characteristic equation;

$\lambda = -1, 1, 2$ are the eigen values.

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the Eigen vector corresponding to Eigen values.

$AX = X\lambda$

$AX - X\lambda = 0$

$(A-\lambda)I = 0$

$$\begin{bmatrix} 1-\lambda & 1 & -2 \\ -1 & 2-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (1).$$

Case 1: If $\lambda = -1 \Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Case 2: If $\lambda = 1 \Rightarrow X_2 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

Case 3: If $\lambda = 2 \Rightarrow X_3 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$

The Modal matrix $B = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$|B| = \begin{vmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} \Rightarrow 1(2-3) - 3(0-3) + 1(0-2)$

$1(-1) - 3(-3) + 1(-2)$

$-1 + 9 - 2 \Rightarrow 6 \neq 0$

$B = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

$\text{Adj}\, B = (\text{cofactor matrix})^T = \begin{bmatrix} 2 & 3 & 0 & 2 \\ 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 2 & 3 & 0 & 2 \end{bmatrix}^T$

$$= \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 2 \\ 7 & -3 & 2 \end{bmatrix}^T$$

$$\text{Adj } B = \begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix}$$

$$B^{-1} = \frac{\text{Adj } B}{|B|} = \frac{1}{6}\begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix}$$

Diagonal Matrix $\Rightarrow D = B^{-1}AB$

$$D = \frac{1}{6}\begin{bmatrix} -1 & -2 & 7 \\ 3 & 0 & -3 \\ -2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}.$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Q2. Diagonalise the matrix A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Sol. Given $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

let λ be the characteristic root or eigen value then characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow 1-\lambda[(5-\lambda)(1-\lambda)-1] - 1[1(1-\lambda)-3] + 3[1-3(5-\lambda)] = 0$

$\Rightarrow 1-\lambda[\lambda^2 - 6\lambda + 4] + \lambda - 1 + 3 + 3 - 45 + 9\lambda = 0$

$\Rightarrow \lambda^2 - 6\lambda + 4 - \lambda^3 + 6\lambda^2 - 4\lambda + \lambda + 6 - 46 + 9\lambda = 0$

$\Rightarrow -\lambda^3 + 7\lambda^2 - 36 = 0 \quad \Rightarrow \lambda^3 - 7\lambda^2 + 36 = 0$ is the characteristic equation.

The Eigen values are $\lambda = -2, 3, 6$.

let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the Eigen vectors corresponding to X.

$$AX = X\lambda$$
$$AX - X\lambda = 0$$
$$(A - \lambda I) = 0$$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

Case (1): If $\lambda = -2$ Substitute in (1).

$$\begin{bmatrix} 3 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} 3x_1 + 1x_2 + 3x_3 &= 0 \\ 1x_1 + 7x_2 + 1x_3 &= 0 \\ 3x_1 + 1x_2 + 3x_3 &= 0 \end{aligned}\right\}$$

$$\begin{matrix} 7 & 1 & 1 & 7 \\ 1 & 3 & 3 & 1 \end{matrix} \Rightarrow \frac{x_1}{21-1} = \frac{x_2}{3-3} = \frac{x_3}{1-21}$$

$$\Rightarrow \frac{x_1}{20} = \frac{x_2}{0} = \frac{x_3}{-20}$$

$$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Case (2): If $\lambda = 3$; Substitute in (1).

$$\begin{bmatrix} -2 & 1 & 3 \\ 1 & 2 & 1 \\ 3 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1+1x_2+3x_3=0$
$1x_1+2x_2+1x_3=0$
$3x_1+1x_2-2x_3=0$

$$\begin{matrix} 2 & 1 & 1 & 2 \\ 1 & -2 & 3 & 1 \end{matrix} \Rightarrow \frac{x_1}{-4-1} = \frac{x_2}{3+2} = \frac{x_3}{1-6}$$

$$\Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$\therefore x_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Case 3 : If $\lambda=6$; Substitute in 1 .

$$\begin{bmatrix} -5 & 1 & 3 \\ 1 & -1 & 1 \\ 3 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-5x_1+1x_2+3x_3=0$
$1x_1-x_2+x_3=0$
$3x_1+1x_2-5x_3=0$

$$\begin{matrix} -1 & 1 & 1 & -1 \\ 1 & -5 & 3 & 1 \end{matrix} \Rightarrow \frac{x_1}{5-1} = \frac{x_2}{3+5} = \frac{x_3}{1+3}$$

$$\Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$$\Rightarrow x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$ $\therefore$ x_1, x_2, x_3 are eigen vectors .

* Check if the vectors are orthogonal or not .

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$x_1 x_2 = x_2 x_3 = x_3 x_1 = 0$$

Modal matrix $= [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$

Normalizing matrix $B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0/\sqrt{2} & -1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

$$B^T = \begin{bmatrix} 1/\sqrt{2} & 0/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

Diagonal Matrix $D = B^T A B$

$$D = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & 2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

//

(7) Reduction of Quadratic form to Canonical form.

Quadratic form : A Homogeneous polynomial of second degree in any number of variables is called quadratic form.

eg: $ax^2+2hxy+by^2$

eg: $ax^2+by^2+cz^2+2hxy+2gyz+2fzx$

Hence every quadratic form can be written as "X^TAX"

So that matrix 'A' is always a 'symmetric matrix'

Now writing the above polynomial in the form of matrix.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Canonical form to Quadratic form :

let "X^TAX" be a Quadratic form then there exist non-singular matrix $D = B^TAB$.

Apply the similarity transformation $X = BY$ to Quadratic form then we get $X^TAX = (BY)^TA(BY)$

$$= Y^TB^TABY \qquad \because (AB)^T = B^TA^T$$

$$\because D = B^TAB.$$

$$= Y^TDY$$

which is canonical form of a given quadratic form.

This form is also called as "Sum of squares form".

Index: No. of positive terms in canonical form.

Signature: Excess no. of positive over negative terms in canonical form.

Q1. Find the symmetric matrix corresponding to the following quadratic form $x^2+2y^2+7z^2+2xy+10yz+6xz$.

Sol. $A=\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 5 \\ 3 & 5 & 7 \end{bmatrix}$

Q2. Write the quadratic form corresponding to the matrix where

$A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Sol. Given $A=\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$x=\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

The Quadratic form; $X^TAX=[x \ y . z]\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= x^2+5y^2+1z^2+2xy+6xz+2yz$ //

Q3. Reduce the following quadratic form to Canonical form or sum of the squares form by orthogonal transformation and also find Rank, Index and Signature.

$3x_1^2+5x_2^2+3x_3^2-2x_1x_2-2x_2x_3+2x_1x_3$

Sol. Given quadratic form is :

$3x_1^2+5x_2^2+3x_3^2-2x_1x_2-2x_2x_3+2x_1x_3$.

$A=\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

let 'λ' be the characteristic root or Eigen value.

The characteristic equation $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow 3-\lambda\left[(5-\lambda)(3-\lambda)-1\right]+1\left[-1(3-\lambda)+1\right]-1\left[1-1(5-\lambda)\right]=0$$

$$\Rightarrow 3-\lambda\left[15-5\lambda-3\lambda+\lambda^2-1\right]+1\left[-3+\lambda+1\right]-1\left[1-5+\lambda\right]=0$$

$$\Rightarrow 3-\lambda\left[\lambda^2-8\lambda+14\right]+1\left[\lambda-2\right]+1\left[\lambda-4\right]=0$$

$$\Rightarrow 3\lambda^2-24\lambda+42-\lambda^3+8\lambda^2-14\lambda+\lambda-2+\lambda-4=0$$

$$\Rightarrow -\lambda^3+11\lambda^2-36\lambda+36=0$$

$\Rightarrow \lambda^3-11\lambda^2+36\lambda-36=0$ is the characteristic equation.

$\lambda=2,3,6$ are the eigen values.

let $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the vector corresponding to λ.

According to Eigen values; $AX=X\lambda$

$(A-\lambda I)X=0$

$$\Rightarrow \begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

Case (1): If $\lambda=2$; Substitute in (1).

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$1x_1 - 1x_2 + 1x_3 = 0$

$$\left.\begin{array}{l} -1x_1 + 3x_2 - 1x_3 = 0 \\ 1x_1 - 1x_2 + 1x_3 = 0 \end{array}\right\}$$

$$\begin{array}{cccc} 3 & -1 & -1 & 3 \\ -1 & 1 & 1 & -1 \end{array}$$

$$\Rightarrow \frac{x_1}{3-1} = \frac{x_2}{-1+1} = \frac{x_3}{1-3}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{0} = \frac{x_3}{\cancel{2}}$$

$$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Case 2 : If $\lambda = 3$; Substitute in 1 .

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} 0x_1 - 1x_2 + 1x_3 = 0 \\ -1x_1 + 2x_2 - 1x_3 = 0 \\ 1x_1 - 1x_2 + 0x_3 = 0 \end{array}\right\}$$

$$\begin{array}{cccc} 2 & -1 & -1 & 2 \\ -1 & 0 & 1 & -1 \end{array} \Rightarrow \frac{x_1}{0-1} = \frac{x_2}{-1-0} = \frac{x_3}{1-2} \Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

$$\Rightarrow X_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Case 3 : If $\lambda = 6$; Substitute in 1 .

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} -3x_1 - x_2 + x_3 = 0 \\ -x_1 - x_2 - x_3 = 0 \\ 1x_1 - x_2 - 3x_3 = 0 \end{array}\right\}$$

$$\begin{array}{cccc} -1 & -1 & -1 & -1 \\ -1 & -3 & 1 & -1 \end{array} \Rightarrow \frac{x_1}{3-1} = \frac{x_2}{-1-3} = \frac{x_3}{1+1}$$

$$\Rightarrow X_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Check orthogonality : $X_1X_2 = X_2X_3 = X_3X_1 = 0$ ✓

Modal Matrix : $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{bmatrix}$

Normalizing Matrix : $B = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0/\sqrt{2} & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$

$$B^T = \begin{bmatrix} 1/\sqrt{2} & 0/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix}$$

Diagonal Matrix : $D = B^TAB$

$$D = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Hence the Transformed matrix - Canonical form is

$$Y^TDY = \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \cancel{3} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$$= 2Y_1^2 + 3Y_2^2 + 6Y_3^2$$

Index : No. of positive terms = 3, Rank = 3 //.

Signature : Positive + Negative = 3+0 = 3, Reduce the quadratic form to Canonical form and also find its Nature (Index, Signature, Rank).

$2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3-2x_3x_1$.

Given quadratic form is:

$2x_1^2+2x_2^2+2x_3^2-2x_1x_2-2x_2x_3-2x_3x_1$

$$A=\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

let 'λ' be the characteristic root or Eigen value.

The characteristic equation $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix}=0$$

$\Rightarrow 2-\lambda\left[(2-\lambda)(2-\lambda)-1\right]+1\left[-1(2-\lambda)-1\right]-1\left[1+1(2-\lambda)\right]=0$

$\Rightarrow 2-\lambda\left[\lambda^2-4\lambda+3\right]+1\left[\lambda-3\right]-1\left[-\lambda+3\right]=0$

$\Rightarrow 2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+\lambda-3-3+\lambda=0$

$\Rightarrow -\lambda^3-9\lambda+6\lambda^2+0=0$

$\Rightarrow \lambda^3-6\lambda^2+9\lambda=0$ is the characteristic equation.

The eigen values are $\lambda=3,0,3$.

let $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the vectors corresponding to eigen values.

According to Eigen values $AX=X\lambda$

$(A-\lambda I)X=0$

$$\begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ — } (1).$$

Case (1): If $\lambda = 3$; Substitute in (1).

$$\Rightarrow \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} -1x_1 - 1x_2 - 1x_3 &= 0 \\ -1x_1 - 1x_2 - 1x_3 &= 0 \\ -1x_1 - 1x_2 - x_3 &= 0 \end{aligned}\right\}$$

$$\begin{matrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{matrix} \Rightarrow \frac{x_1}{1-1} = \frac{x_2}{1-1} = \frac{x_3}{1-1}$$

$$\Rightarrow \frac{x_1}{0} = \frac{x_2}{0} = \frac{x_3}{0}$$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

let $x_3 = \cancel{k_2}\ k_1$, $x_2 = k_2$

$-1x_1 - x_2 - x_3 = 0$

$-x_1 = x_2 + x_3$

$x_1 = -x_2 - x_3 = -k_2 - k_1$

$x_2 = k_2$ $\quad$ $x_1 = -k_2 - k_1$

$x_3 = k_1$

linear combination;

$k_1x_1 + k_2x_2 + k_3x_3 = 0$

$k_1\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} + k_2\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

$$\therefore x_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

∴ Eigen values = 3, 0, 3

∴ Eigen vectors $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Case ②: If $\lambda = 0$; Substitute in ①.

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$\Rightarrow \left.\begin{array}{l} 2x_1 - 1x_2 - 1x_3 = 0 \\ -1x_1 + 2x_2 - 1x_3 = 0 \\ -1x_1 - 1x_2 + 2x_3 = 0 \end{array}\right\}$$

$$\begin{array}{cccc} 2 & -1 & -1 & 2 \\ -1 & 2 & -1 & -1 \end{array} \Rightarrow \frac{x_1}{4-1} = \frac{x_2}{1+2} = \frac{x_3}{1+2}$$

$$\Rightarrow \frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3} \Rightarrow X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The Eigen values are 3, 0, 3.

The Eigen vectors are $X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Check orthogonality : $X_1 X_2 \neq 0$

$X_2 X_3 = X_3 X_1 = 0$.

To get X_2 orthogonal to X_1 ; assume a vector $[u, v, w]'$ orthogonal to X_1 also satisfying

$-1x_1 - 0x_2 + 1x_3 = 0$

i.e $-1 \cdot u + 1 \cdot v + 0 \cdot w = 0$, $u + v + w = 0$.

Solving $[u, v, w]'$ $\Rightarrow$ $-u + w = 0 \Rightarrow w = u = 0$

$-u + v = 0 \Rightarrow v = u = 0$

$\therefore x_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\therefore X_1 X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $X_2 X_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$X_3 X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow X_1 X_2 = X_2 X_3 = X_3 X_1$.

$\therefore$ Modal Matrix: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

Normalizing matrix $B = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0/\sqrt{2} & 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix}$

$B^T = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

Diagonal Matrix : $\boxed{D = B^T A B}$

$D = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & 0 & 1/\sqrt{3} \end{bmatrix}$

$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Hence the Transformed Canonical form is $Y^T D Y$.

$$\Rightarrow \begin{bmatrix} Y_1 & Y_2 & Y_3 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

$= 3Y_1^2 + 3Y_2^2 + 0Y_3^2$ is the canonical form.

$\therefore$ Index : Number of positive terms $= 2$.

Signature : Number of positive + Negative terms $= 2+0 = 2$

Rank : Number of non zero rows $= 2$.